CHAPTER 2

Limits and the Derivative

I do not know what I may appear to the world, but to myself I seem to have been only a boy playing on the sea-shore, and diverting myself in now and then finding a smoother pebble or a prettier shell than ordinary, whilst the great ocean of truth lay all undiscovered before me.

SIR ISAAC NEWTON (1642–1727)

Taking mathematics from the beginning of the world to the time of Newton, what he has done is much the better half.

GOTTFRIED WILHELM VON LEIBNIZ (1646–1716)
(generally credited with having created calculus independently of Newton)

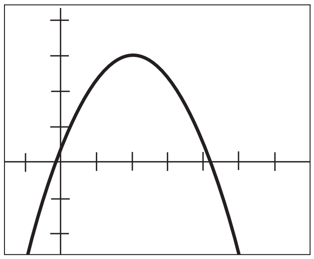
Nature and Nature's laws lay hid in night:
God said, Let Newton be! and all was light.

ALEXANDER POPE (1688–1744)

CALCULATOR CALCULUS

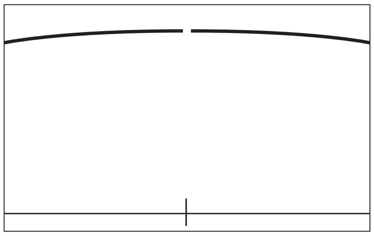
Imagine that you are a video game designer and programmer. (This is a career that requires mathematics—do some research and learn more about it!) You want to design an environment filled with hidden perils for unwary players. You decide to give each character the ability to visually zoom in on any part on this environment—the player will get points if the character spots, and avoids, a pitfall.

The graph below might represent one small portion of a two-dimensional environment. (It may also remind you of the "hills" you modeled in Chapter 1!)



Suppose a character is "climbing the hill." Toward what point is she headed? (Look for the coordinates of the "top of the hill.")

But, the character is unwise and fails to zoom in on the summit. If she did, she might spot the tiny gap:



You might imagine what will happen when she reaches the top—the ground disappears and she plummets! Would you still answer the question "toward what point is she heading" the same way? After all, she doesn't know she is doomed to tumble through a hole. Her steps still lead toward the point (2, 3).

In calculus we will make a big deal out of the distinction between a function value (or the absence of one) and the intuitive concept of "headed toward." (This concept will be made much more explicit in the chapter.)

Project Idea:

Put some "holes" in the hills you modeled in Chapter 1 (see chapter opener.) You can do this by replacing your "hill" model, say $f(x) = -x^2 + 1$, with a rational function

$$g(x) = \frac{f(x)h(x)}{h(x)}$$

Let $h(x)$ be any linear function. Try several, making sketches as you go. Then answer the following questions:

1. How does the location of the hole depend upon $h(x)$?
2. When do the output values of $y = f(x)$ differ from those of $y = g(x)$?

As you proceed through this chapter, keep this example in mind.

■ ■ ■

CALCULUS HAS ITS origin in several related problems. Two of these are the *problem of tangents* and the determination of the *velocity of a moving object.* We begin this chapter with a discussion of each of these questions (which turn out to be equivalent). The results lead to one of the fundamental ideas of calculus, that of the *derivative.* In order to discuss the derivative, however, we need the concept of *limit*, which is intuitively involved in the problems of tangents and velocity. The careful development of this concept is sometimes regarded as a digression by students. In fact, however, an understanding of limits is basic to the study of calculus.

2.1 Two Equivalent Problems

Tangent to a Curve

We know from plane geometry that the tangent to a circle at a given point is the straight line through the point perpendicular to the radius drawn to the point. See Figure 1, in which we show the circle $x^2 + y^2 = 25$ and the tangent at $(-3, 4)$. Since the slope of the radius is $-\frac{4}{3}$, the slope of the tangent is $\frac{3}{4}$ and hence its equation is $y - 4 = \frac{3}{4}(x + 3)$, or $3x - 4y + 25 = 0$.

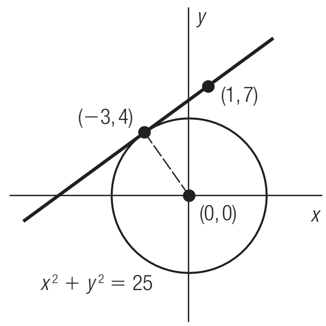


Figure 1 Tangent to a circle

Things are not so simple when the given curve is not a circle. Sometimes (by analogy with the circle) it is suggested that a tangent is a "straight line making contact with the curve at exactly one point," but you can see from Figure 2 that such a definition is inadequate. Nevertheless most people have an intuitive idea of what a tangent is; the problem is to be precise about it.

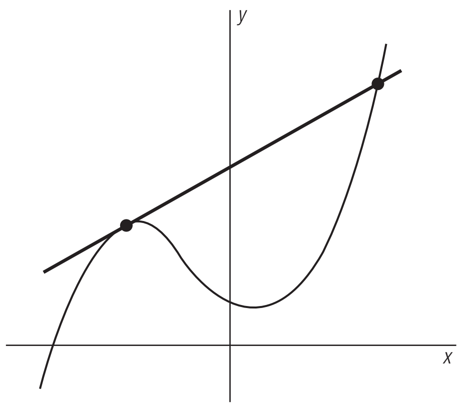


Figure 2 Tangent making contact with a curve more than once

■ **Example 1**

Discuss the problem of finding the tangent to the curve $y = x^2$ at the point $(1,1)$.

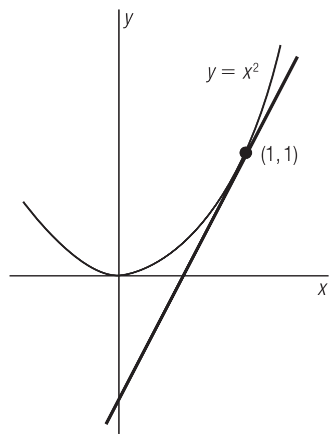


Figure 3 Tangent to $y = x^2$ at $(1,1)$

Solution

The graph, together with the tangent at $(1,1)$, is shown in Figure 3. Of course that begs the question; it is hardly fair to draw it before we know what it is! Nevertheless,

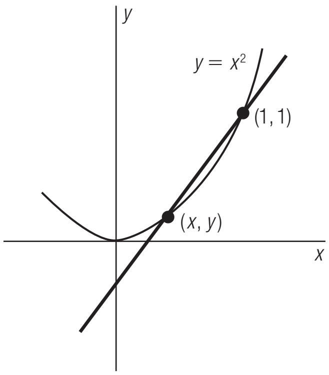


Figure 4 Approximation to tangent at (1, 1)

we are going to begin by assuming that a definite tangent exists, our plan being to sneak up on it.

There is no difficulty in drawing a line through (1, 1) and a neighboring point (x, y) on the parabola. (See Figure 4.) This line is not the tangent, but it is a good substitute if (x, y) is near (1, 1). Its slope is

$$Q(x) = \frac{y-1}{x-1} = \frac{x^2-1}{x-1}$$

where we use the functional notation $Q(x)$ to indicate that the slope is a quotient whose value depends on x. Note that $Q(x)$ is defined only for values of $x \neq 1$, since (x, y) must be distinct from (1, 1) if we are to draw the line shown in Figure 4.

Now imagine the point (x, y) brought closer to the point (1, 1). The nearer it gets, the closer x is to 1. The question is, how does the slope $Q(x)$ behave during this process? Observe that for all $x \neq 1$ we can write

$$Q(x) = \frac{(x-1)(x+1)}{x-1} = x + 1$$

and from this it is evident that as x gets closer to 1, the slope approaches 2. While x cannot be allowed to *equal* 1 (and hence the slope never equals 2), there is nevertheless no doubt about the number $Q(x)$ is approaching. It is not, for example, approaching 3, or 2.01, or 1.9997; it is approaching 2. Why not agree that the tangent is the line through (1, 1) with slope 2? No other agreement would be sensible, so we adopt this as the *definition* of the tangent.

Thus the problem of finding the tangent at (1, 1) is solved. The line through (1, 1) with slope 2 is represented by the equation $y - 1 = 2(x - 1)$ or $2x - y - 1 = 0$. In Figure 3 it may appear that we had to guess the direction of the line, but now we can plot two of its points, say (1, 1) and (0, −1), and sketch it accurately. ■

Remark

In Example 1 the reader has a right to object that we began by evading the question of what a tangent is. We said that we would sneak up on it, but what line did we actually have in mind? One answer is the line through (1, 1) with the same direction as the curve at that point, but what does "direction" mean? Is it our line of sight as we move along the curve looking straight ahead? That is simply the tangent!

The difficulty is that we began by talking about a concept that is intuitively clear but lacks precise definition. We used our intuition to lead us to the definition, but once the definition is adopted we don't need intuition. We can appeal to the definition instead and say, "That is the line we had in mind." Study Example 1 carefully to appreciate this point. What looks like circular reasoning is really motivation for a definition. If we were merely trying to be logical, we would give the definition and be done with it. (Definitions require no introduction or defense.) We would say, "The tangent to the curve $y = x^2$ at the point (1, 1) is the line through (1, 1) with slope 2." But then you might justly accuse us of being arbitrary. In the end *we are*; every definition is arbitrary! Mathematicians, however, are no different from other people in their desire to be understood.

Note that in the end there is no mystery about direction. When you reach (1,1) in your travel along the curve, simply look ahead along the line $2x - y - 1 = 0$ we found in Example 1. More precisely, define the **slope of the curve** (previously an ambiguous idea) to be the slope of the tangent, namely 2.

■ Example 2

The slope of the curve $y = x^2$ at (1,1) is 2, as we agreed in Example 1. There is no reason why we cannot adopt a similar definition at any point of the curve, say (x_0, y_0). If (x, y) is a neighboring point of the curve, the line through (x_0, y_0) and (x, y) is an approximation to the tangent. Its slope is

$$Q(x) = \frac{y - y_0}{x - x_0} = \frac{x^2 - {x_0}^2}{x - x_0} = \frac{(x - x_0)(x + x_0)}{x - x_0} = x + x_0 \qquad (x \neq x_0)$$

Closeness of (x, y) to (x_0, y_0) implies closeness of x to x_0, which implies that $Q(x)$ is nearly equal to $2x_0$. Hence we define the tangent at (x_0, y_0) to be the line through (x_0, y_0) with slope $2x_0$. An equation of the tangent is

$$y - y_0 = 2x_0(x - x_0)$$

Thus the slope of the curve $y = x^2$ at any point (x_0, y_0) is $2x_0$. To say the same thing without subscripts, the slope of the curve $y = x^2$ at any point (x, y) is $m(x) = 2x$. The slope depends on x. At (1,1) it is 2; at (0,0) it is 0; at (2,4) it is 4; and so on. ■

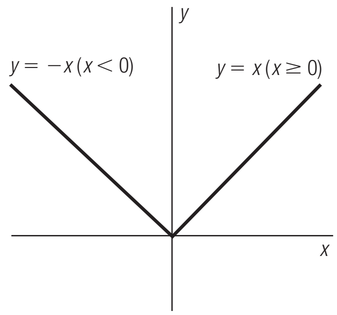


Figure 5 Graph of $y = |x|$

■ Example 3

The graph of

$$y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

consists of two rays meeting at the origin, as shown in Figure 5. Discuss its slope at the origin.

Solution

There is a "corner" at (0,0), which suggests that we will have difficulty defining a tangent at that point. To find out what happens, let (x, y) be a neighboring point of the graph. The slope of the line through (0,0) and (x, y) is

$$Q(x) = \frac{y - 0}{x - 0} = \frac{y}{x} = \frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$

(Note that x cannot be 0. Why?) When (x, y) is close to (0,0), what is $Q(x)$ close to? It is 1 or -1 depending on whether $x > 0$ or $x < 0$. There is no number m we can name that $Q(x)$ approaches as x approaches 0.

We describe this situation by saying that there is no tangent at (0,0). Nor does the graph have a slope at that point. A traveler moving from left to right on this path would be puzzled if you asked about the direction of the path at (0,0). (See

Figure 6.) The traveler might say, "Well, I was going southeast. But this is ridiculous. I had to come to a screeching halt and now apparently I am about to go northeast. The least they could do is put up some signs." ■

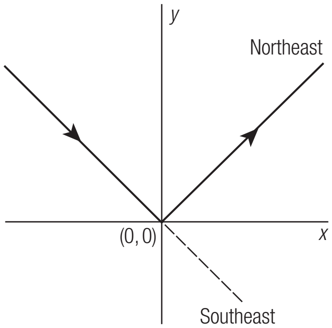


Figure 6 What is the direction at (0,0)?

■ Example 4

The graph of $y = \sqrt{x}$ is the upper half of the parabola $y^2 = x$ (Figure 7). It is geometrically apparent that its tangent at the origin is the y axis, which leads us to believe that the slope at (0,0) is undefined. We confirm this by looking at

$$Q(x) = \frac{y - 0}{x - 0} = \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}} \qquad (x > 0)$$

When x approaches 0, $Q(x)$ increases without bound, which is what we should expect. This means that the line through (0,0) and (x,y) gets steeper as (x,y) approaches the origin. The line it approaches is the vertical line $x = 0$. Thus in this case the curve has a tangent, but its slope is undefined. ■

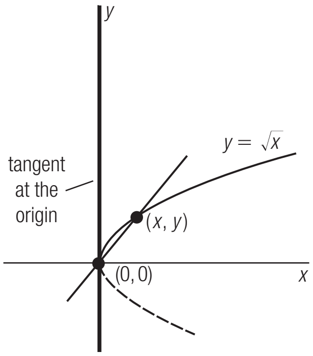


Figure 7 Vertical tangent at (0,0)

Velocity of Motion in a Line

Now we turn to a question that is closely related to the problem of tangents. Suppose that a ball is thrown straight upward, its height above the ground t seconds later being $h = 64t - 16t^2$ (in feet). The source of this formula need not concern us now; we simply take it as given. What we want to discuss is the *velocity* of the ball.

Presumably you already know how to compute an *average rate*, as in the case of a car making a trip of 150 miles in 3 hours. When we say that its average rate is 50 mph we are using the formula

$$\text{Rate} = \frac{\text{Distance}}{\text{Time}}$$

(sometimes written in the form Distance = Rate × Time). This formula is not much help in the case of a ball thrown upward from the ground because the rate is not constant. Moreover, the ball reverses direction at its highest point; our discussion of velocity should include a way of distinguishing between upward and downward motion.

What we need is a mathematical refinement of the operation of a radar unit, which measures the *instantaneous rate* at which an object is moving. The reason it works so well is that the radar pulse (from the unit to the moving object and back) travels at the speed of light. Hence the object moves a very short distance in a very small interval of time while its rate is being measured.

Let's apply that idea to the ball. When $t = t_0$, the height of the ball is $h_0 = 64t_0 - 16t_0^2$. If t is a later clock reading $(t > t_0)$, the height has become $h = 64t - 16t^2$. The *change in position* (called "displacement") is

$$h - h_0 = (64t - 16t^2) - (64t_0 - 16t_0^2) = 64(t - t_0) - 16(t^2 - t_0^2)$$

The corresponding change in time is $t - t_0$; we define the *average velocity* during the time interval $[t_0, t]$ to be

$$\frac{h-h_0}{t-t_0}=\frac{64(t-t_0)-16(t^2-t_0^{\,2})}{t-t_0}=64-16(t+t_0)$$

This is the same idea as our computation of average rate in the case of a car going 150 miles in 3 hours, but since it allows for negative (or zero) displacement we call it *velocity*. (*Speed* is the absolute value of velocity.)

Remark

If the ball is rising during the time interval $[t_0, t]$, displacement is the same as distance traveled, and the average velocity is positive. In general, however, it may be positive, negative, or even zero. If $t_0 = 1.9$, for example, and $t = 2.1$, then $h_0 = 63.84$ and $h = 63.84$, so the average velocity is

$$\frac{h-h_0}{t-t_0}=\frac{0}{0.2}=0$$

This does not mean that the ball is motionless; instead it rises from $h_0 = 63.84$ to its highest point and then falls back to $h = 63.84$.

It is also worth noting that there is no mathematical reason to restrict $t > t_0$. If we allow $t < t_0$, the time interval is $[t, t_0]$ instead of $[t_0, t]$ and the average velocity is

$$\frac{(\text{terminal position})-(\text{initial position})}{(\text{later time})-(\text{earlier time})}=\frac{h_0-h}{t_0-t}=\frac{h-h_0}{t-t_0}$$

The ratio is the same either way.

The next step should be apparent. To make the average velocity a good approximation to the *instantaneous velocity* at time t_0, we choose t close to t_0 (as is done in a radar unit). More precisely, we evaluate the *limit* of average velocity as t approaches t_0. As you can see from our formula for average velocity, the limit is

In using the term "limit" here, we are leaning on your intuition. The term will be carefully defined in the next section.

$$v_0 = 64 - 16(t_0 + t_0) = 64 - 32t_0$$

Dropping the subscript (the only purpose of which was to distinguish the fixed instant t_0 from the variable time t), we have a formula for instantaneous velocity v at time t, namely

$$v = 64 - 32t = 32(2 - t)$$

This formula is a precise instrument for discussing the motion of the ball, for it tells us not only *how fast* the ball is moving at any time t, but also *in what direction*. The following table of heights and velocities illustrates what we mean.

t	0	1	2	3	4
h	0	48	64	48	0
v	64	32	0	−32	−64

When $t = 0$ the ball is leaving the ground ($h = 0$) with speed 64 ft/sec, moving upward. One second later it has reached a height of 48 ft and is still moving upward with speed 32 ft/sec. At $t = 2$ it has reached its highest point (64 ft) and has come to a momentary stop. When $t = 3$ the ball is back to height $h = 48$ and is

GRAPHING CALCULATOR CONCEPTS

Tables

You can reproduce the time-height-velocity table given in the text using your calculator's *table* feature. Simply set the height formula $y_1 = 64x - 16x^2$ and the velocity formula $y_2 = 64 - 32x$ (note that you must replace the more intuitively named variables t, h, and v with x and y.) Now, being sure to set the table properties to allow you to select your own values for the input variables, look at the table and choose the x values 0, 1, 2, 3, and 4. You can go between these values to refine the table.

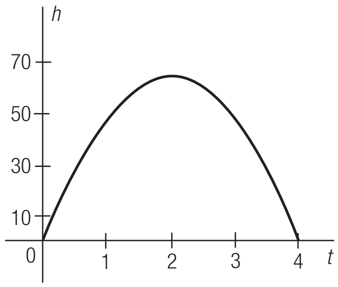


Figure 8 Graph of $h = 64t - 16t^2, 0 \le t \le 4$

falling with speed 32 ft/sec (because the velocity is negative). It strikes the ground when $t = 4$ with speed 64 ft/sec (the same speed with which it was thrown, but the motion is opposite in direction).

A graph of h as a function of t is shown in Figure 8 (not to be confused with the path of the ball, which is in a vertical straight line). The formula $v = 64 - 32t$ is nothing more than the slope of this graph, as we can check by the methods described in the first part of this section. For if (t_0, h_0) is an arbitrary point of the graph and (t, h) is a nearby point, the slope of the line joining them is

$$Q(t) = \frac{h - h_0}{t - t_0} = 64 - 16(t + t_0)$$

(the average velocity computed earlier). Its limit as t approaches t_0 is the slope of the graph at (t_0, h_0), namely $m(t_0) = 64 - 32t_0$. The slope at any point (t, h) is therefore $m(t) = 64 - 32t$, which is the formula for velocity at time t.

■ Example 5

Suppose that an object is moving along a coordinate line (the s axis) in such a way that its position at time t is $s = p(t) = t^2$. Find a formula for velocity.

Solution

Let t_0 be the instant at which we are going to compute velocity and suppose that $t \neq t_0$. The average velocity during the time interval with endpoints t_0 and t is

$$Q(t) = \frac{\text{change in position}}{\text{change in time}} = \frac{p(t) - p(t_0)}{t - t_0}$$

$$= \frac{t^2 - t_0^2}{t - t_0} = \frac{(t - t_0)(t + t_0)}{t - t_0} = t + t_0$$

When t approaches t_0, $Q(t)$ approaches $2t_0$, so the velocity at t_0 is $v(t_0) = 2t_0$. The same statement without subscripts is that the velocity at time t is $v(t) = 2t$. ■

This derivation of $v(t) = 2t$ from the law of motion $s = t^2$ is mathematically indistinguishable from Example 2, where we derived the slope $m(x) = 2x$ from the equation $y = x^2$. We may use that fact to interpret the meaning of positive and negative velocity. When $t = -1$, for example, the velocity is $v = -2$. The negative clock reading is no problem; it is like the year 1 B.C. (simply an earlier time than the one we choose to call 0). But what are we to make of negative velocity?

Recall from Section 1.2 that negative *slope* (of a straight line) means that the line is falling from left to right. If the equation of the line is $y = mx + b$, this implies that y is decreasing as x increases. It is geometrically apparent (somewhat harder to prove!) that negative slope of a *curve* means the same thing. Thus the parabola $y = x^2$ is falling from left to right as we pass through the point $(-1,1)$ because the slope is negative ($m = -2$ at $x = -1$). (See Figure 9.) The same statement about the law of motion $s = t^2$ is that s is decreasing as the clock ticks off $t = -1$ because the velocity is negative ($v = -2$ at $t = -1$). Since s is the coordinate of an object moving along the s axis, the object must be going in the negative direction of that axis. Similarly, positive velocity means that the object is moving in the positive direction of the s axis.

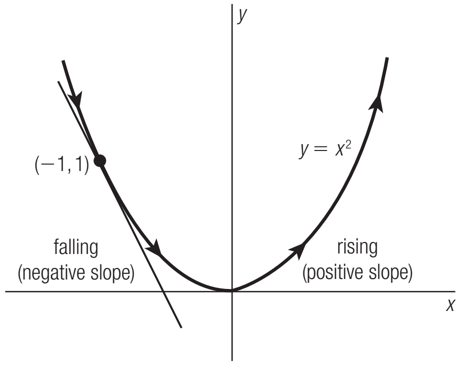


Figure 9 Interpretation of slope

We may summarize as follows:

> Suppose that an object is moving along a coordinate line with velocity v at time t. Its **speed** is the absolute value of v. Its **direction** is positive if $v > 0$ and negative if $v < 0$. (If $v = 0$, it has come to a momentary stop.)

When we get to the subject of motion in a curve, you should remember what we have said about velocity here. Its two qualities of speed and direction (which in linear motion require nothing more than a number with a sign) will be described by using *vectors*. The purpose of this section, however, is merely to exhibit the equivalence of the problem of tangents and the problem of velocity, in preparation for the definition of derivative in Section 2.4.

Problem Set 2.1

1. Let $Q(x)$ be the slope of the line through $(2,4)$ and a neighboring point (x,y) of the parabola $y = x^2$.
   (a) What is the formula defining Q? For what values of x is $Q(x)$ defined?
   (b) What number does $Q(x)$ approach as (x,y) approaches $(2,4)$?
   (c) Find an equation of the tangent at $(2,4)$.
2. Let $Q(x)$ be the slope of the line through $(1,-1)$ and a neighboring point (x,y) of the parabola $y = -x^2$.
   (a) What is the formula defining Q? For what values of x is $Q(x)$ defined?
   (b) What number does $Q(x)$ approach as (x,y) approaches $(1,-1)$?
   (c) Find an equation of the tangent at $(1,-1)$.

In each of the following, find the slope of the graph at the given point and an equation of the tangent at that point.

3. $y = 3x^2$ at $(2,12)$
4. $y = 1 - x^2$ at $(0,1)$ How could the result have been predicted graphically?
5. $y = \frac{1}{2}x^2$ at $(0,0)$ How could the result have been predicted graphically?
6. $y = x^3$ at $(1,1)$ *Hint*: $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$
7. $y = \sqrt{x}$ at $(1,1)$ *Hint*: $x - 1 = (\sqrt{x} - 1)(\sqrt{x} + 1)$
8. $y = x|x|$ at $(0,0)$ How could the result have been predicted graphically?

In each of the following, s is the position of a moving object at time t. Find the velocity at the given instant.

9. $s = t^2 + 1$ at 1
10. $s = t^2 - 1$ at 1
11. $s = t^3 - 4$ at 2
12. $s = \frac{1}{2}t^3$ at 2
13. $s = 1/t$ at 2
14. $s = \sqrt{t}$ at 4
15. If (x_0, y_0) is a point of the curve $y = 3x^2$, let $Q(x)$ be the slope of the line through (x_0, y_0) and a neighboring point (x, y) of the curve.
    (a) What is the formula defining Q? For what values of x is $Q(x)$ defined?
    (b) Find the slope of the curve at (x_0, y_0) by determining what $Q(x)$ approaches as (x, y) approaches (x_0, y_0).
    (c) Drop the subscripts in the answer to part (b) to obtain the slope at any point (x, y). Does the result check with Problem 3 when $x = 2$?

In each of the following, find the slope of the graph at (x_0, y_0). Then drop the subscripts to obtain the slope at any point (x, y).

16. $y = 1 - x^2$ (Compare with Problem 4.)
17. $y = \frac{1}{2}x^2$ (Compare with Problem 5.)
18. $y = x^3$ (Compare with Problem 6.)
19. $y = \sqrt{x}$ For what values of x is the slope at (x, y) defined?

In each of the following, a law of motion is given. Find the velocity at time t_0, then drop the subscript to obtain the velocity at time t. (The result should check with the corresponding special case in Problems 9 through 14.)

20. $s = t^2 + 1$
21. $s = t^2 - 1$
22. $s = t^3 - 4$
23. $s = \frac{1}{2}t^3$
24. $s = 1/t$
25. $s = \sqrt{t}$

In each of the following, use formulas derived in Problems 15 through 19 to determine where the graph is rising (positive slope), where it is falling (negative slope), and where it flattens out (zero slope). Sketch the graph using this information.

26. $y = 3x^2$
27. $y = 1 - x^2$
28. $y = \frac{1}{2}x^2$
29. $y = x^3$
30. $y = \sqrt{x}$

In each of the following, use formulas derived in Problems 20 through 25 to determine when the object is moving in the positive direction, when it is moving in the negative direction, and when it comes to a momentary stop.

31. $s = t^2 + 1$
32. $s = t^2 - 1$
33. $s = t^3 - 4$
34. $s = \frac{1}{2}t^3$
35. $s = 1/t$ (Assume that $t > 0$.)
36. $s = \sqrt{t}$
37. An object dropped near the surface of the earth (and encountering no air resistance) falls a distance $s = \frac{1}{2}gt^2$ in t seconds (where g is a constant). Show that the velocity of the object t seconds after it is dropped is $v(t) = gt$.
38. A stone is thrown straight upward. After t seconds its height above the ground (in feet) is $s = 32t - 16t^2$.
    (a) Show that the velocity of the stone at time t is $v(t) = 32 - 32t$. What is its initial velocity?
    (b) When does the stone reach its highest point and how high does it rise?
    (c) When does the stone return to the ground and what is its velocity when it hits the ground? What is its speed at that instant?
39. A ball is thrown straight upward from the top of a building. After t seconds its height above the ground (in feet) is $s = 96 + 16t - 16t^2$.
    (a) How tall is the building?
    (b) Show that the velocity of the ball at time t is $v(t) = 16 - 32t$. What is its initial velocity?
    (c) When does the ball reach its highest point and how high (above the ground) does it rise?
    (d) Assuming that the ball returns to the roof of the building, find when it lands. What is its velocity at that instant? its speed?
    (e) The ball could have been thrown by a person whose arm was extended beyond the edge of the roof. In that case it would land on the ground. When would it land and with what velocity? with what speed?
40. Show that the slope of the graph of $y = x^3 - 3x$ at (x, y) is $m(x) = 3x^2 - 3$. Then confirm that there are turning points at $x = \pm 1$ (See Problem 19, Section 1.5.)
41. Show that the slope of the graph of $y = x + (1/x)$ at (x, y) is $m(x) = 1 - (1/x^2)$. Use the result to confirm that there are turning points at $x = \pm 1$. (See Problem 27, Section 1.5.)

42. Explain why the tangent to the curve $y = x^{1/3}$ at the origin is vertical. Sketch the graph. *Hint*: The curve is symmetric about the origin. (Why?)

43. Explain why the graph of $y = 1 - x^{2/3}$ has a vertical tangent at $(0, 1)$. Sketch the graph. *Hint*: The curve is symmetric about the y axis and $(0, 1)$ is its highest point. (Why?)

44. Show that if $x_0 > 0$, the slope of the curve $y = x^{3/2}$ at (x_0, y_0) is $\frac{3}{2}\sqrt{x_0}$. *Hint*: To simplify the formula for slope of the line through (x_0, y_0) and (x, y), let $a = x_0^{1/2}$ and $b = x^{1/2}$. The formula becomes

$$Q(x) = (b^3 - a^3)/(b^2 - a^2)$$

45. From Problem 44 we may conclude that the slope of the graph of $y = x^{3/2}$ at (x, y) is $m(x) = \frac{3}{2}\sqrt{x}$ $(x > 0)$. Why does this formula also apply when $x = 0$? Sketch the graph of $y = x^{3/2}$ by noting where it rises, falls, and flattens out.

46. Show that if the law of motion of a moving object is quadratic ($s = at^2 + bt + c$), the velocity is linear, $v(t) = 2at + b$.

47. In Problem 46, let t_1 and t_2 be endpoints of a time interval and let t be its midpoint. Show that the velocity at time t is the average of the velocities at t_1 and t_2.

48. In Problem 47, show that the velocity at the midpoint is also the average velocity during the interval. (Thus in a law of motion $s = at^2 + bt + c$, the "average velocity" as defined in the text can be computed by averaging the instantaneous velocities at the endpoints of the interval.)

49. Give an example of a law of motion for which neither of the statements in Problems 47 and 48 is true.

50. Suppose that the periodic motion of an object bobbing up and down at the end of a spring is represented by $s = \sin t$.
    (a) Where is the object when $t = 0, \pi/2, \pi, 3\pi/2, 2\pi$?
    (b) Show that the average velocity during the time interval with endpoints t and $t + h$ $(h \neq 0)$ is

    $$\sin t \left(\frac{\cos h - 1}{h}\right) + \cos t \left(\frac{\sin h}{h}\right)$$

    Hint: Use the addition formula

    $$\sin(u + v) = \sin u \cos v + \cos u \sin v$$

    from trigonometry.
    (c) It can be shown that as h approaches 0 the expressions $(\cos h - 1)/h$ and $(\sin h)/h$ approach 0 and 1, respectively. Use that fact to find the velocity at time t.
    (d) How fast, and in what direction, is the object going when $t = 0, \pi/2, \pi, 3\pi/2, 2\pi$?

51. What do you think the graph of $f(x) = \sqrt{1 - x^{2/3}}$ looks like? How does it compare to $g(x) = \sqrt{1 + x^{2/3}}$? Use your graphing calculator to examine these graphs; set the viewing window to $-2 \le x \le 2$, $-2 \le y \le 2$. Use the drawing feature to draw several tangents. Explain how the curves rise and fall. What seems to be happening to the slope of $f(x)$ as x approaches 1 $(x < 1)$?

2.2 Limits

In Section 2.1 (Example 1) we agreed to find the slope of the curve $y = x^2$ at $(1, 1)$ by examining the quotient

$$Q(x) = \frac{y - 1}{x - 1} = \frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{x - 1}$$
$$= x + 1 \quad (x \neq 1)$$

We said that $Q(x)$, while undefined at $x = 1$, approaches 2 as x approaches 1, and we adopted the number $m = 2$ as the answer to the question.

The statement "$Q(x)$ approaches 2 as x approaches 1" may be interpreted geometrically by looking at the graph of Q. Since $Q(1)$ is undefined, while $Q(x) = x + 1$ for all $x \neq 1$, the graph is a straight line with a hole in it. (See Figure 1.) The coordinates of the hole are $(1, 2)$, so the y coordinate of a bug traveling on the graph gets

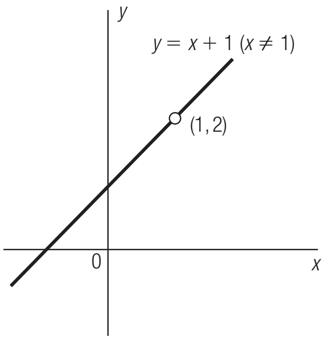


Figure 1 Graph of $Q(x) = \dfrac{x^2 - 1}{x - 1}$

closer to 2 as its x coordinate approaches 1. We express this in somewhat different language by saying that "the limit of $Q(x)$ as x approaches 1 is 2" or (in symbolic form)

$$\lim_{x \to 1} Q(x) = 2$$

This statement cannot be interpreted as an evaluation of $Q(x)$ at $x = 1$. While it is true that $Q(x) = x + 1$, and this formula yields 2 when x is replaced by 1, it is nevertheless meaningless to say that $Q(1) = 2$. $Q(x)$ is the slope of the line through $(1, 1)$ and (x, y), which is not defined unless (x, y) and $(1, 1)$ are distinct. Thus the statement

$$\lim_{x \to 1} Q(x) = 2$$

does not refer to what happens at $x = 1$ but to the behavior of $Q(x)$ when x is near 1.

Perhaps this is clear enough, at least on intuitive grounds. However, there are difficulties.

GRAPHING CALCULATOR CONCEPTS

Graphing "Holes"

Can you find the "gap" (the hole) in the function in Figure 1, using your graphing calculator? Your calculator draws a graph by plotting many points. It doesn't "know" that there is a hole at (1, 2), but it does leave a gap in the line at that point. You will be able to see it if you reset the viewing window to $0 \le x \le 2$, $1 \le y \le 3$.

Try to find the hole in the graph of

$$y = \frac{x^2 - 4x}{x - 4}$$

Write two compound inequalities to describe the viewing window you used.

■ Example 1(a)

We have argued that if

$$Q(x) = \frac{x^2 - 1}{x - 1}$$

then

$$\lim_{x \to 1} Q(x) = 2$$

our reasoning being that when x is close to 1, $Q(x) = x + 1$ is close to 2. But suppose a critic suggests that

$$\lim_{x \to 1} Q(x) = 2.001$$

arguing that when x is close to 1, $Q(x)$ is near 2.001. On what grounds do we say that the critic is wrong?

While this may seem to be a perverse question, it is a difficulty we must meet and overcome. No mathematical concept is useful if it is so imprecise as to allow two people to come up with different answers. Perhaps we do not think of 2 and 2.001 as very far apart, but they are nevertheless distinct. We cannot afford to disagree at all; the limit is either 2 or it isn't, and we must settle on some definitive way to reach a decision.

Mathematicians struggled for a long time to develop the definition we are going to give, but it is simple enough as it applies to this example. The idea is that if 2 is the answer, we should be able to force $Q(x)$ *as close to 2 as our critic desires.* (Closer, for example, than 2.001, if required.) So we let our critic define "close." And we remember that $Q(x)$ depends on x; we control its behavior by placing restrictions on x.

It is like a contest. Our critic goes first, naming a neighborhood of 2 in which $Q(x)$ is to lie. We go second, responding to the challenge by naming a neighborhood of 1 in which x should lie in order to keep $Q(x)$ where our critic wants it. *If we can respond to every challenge*, our critic must agree that the answer is 2. *If there is* *even one neighborhood of 2 to which we are unable to confine $Q(x)$ by keeping x close to 1*, however, we must admit that the answer is not 2.

To understand the definition we draw from this process, it is important to realize that a **neighborhood** of a point p is an *open interval containing p*. (See Figure 2.) If p is deleted, the neighborhood is said to be **punctured**. Occasionally we will also refer to a **right neighborhood** of p and a **left neighborhood** of p, which are open intervals of the type (p, b) and (a, p), respectively.

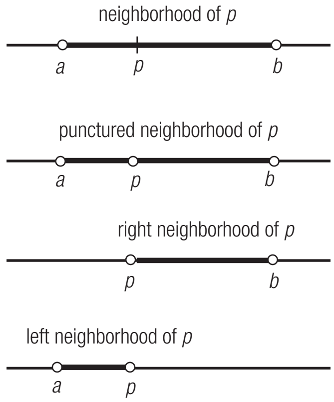


Figure 2 Neighborhoods of a point

Our definition says that

$$\lim_{x\to 1} Q(x) = 2$$

provided that the following condition is met:

> Given any neighborhood of 2 (say N), there is a punctured neighborhood of 1 (say M) with the property that $x \in M \Rightarrow Q(x) \in N$.

The reason we puncture the neighborhood of 1 is that the domain of Q excludes 1. The implication

$$x \in M \Rightarrow Q(x) \in N$$

would fail if we allowed $x = 1$. ■

■ Example 1(b)

Suppose that our critic wants $Q(x)$ to lie in the interval $N = (1.98, 2.05)$, that is $1.98 < Q(x) < 2.05$. To discover how we should restrict x in response to this challenge, we solve the inequalities

$$1.98 < \frac{x^2 - 1}{x - 1} < 2.05$$

that is, $1.98 < x + 1 < 2.05$ $(x \neq 1)$. Subtracting 1 from each side, we find

$$0.98 < x < 1.05 \qquad (x \neq 1)$$

which means that we may choose M to be the interval $(0.98, 1.05)$ with 1 deleted. Since

$$\begin{aligned} x \in M &\Rightarrow 0.98 < x < 1.05 && (x \neq 1) \\ &\Rightarrow 1.98 < x + 1 < 2.05 && (x \neq 1) \\ &\Rightarrow Q(x) \in N \end{aligned}$$

our critic should be satisfied. ■

■ Example 1(c)

We must convince ourselves that our critic cannot baffle us by *any* challenge. Instead of $N = (1.98, 2.05)$ suppose that an arbitrary neighborhood of 2 is named. It ought to be apparent that we can assume this neighborhood is symmetric about 2, simply by using the closer endpoint to compute the radius. For example, there should be no complaint if we replace $(1.98, 2.05)$ by $(1.98, 2.02)$, which has midpoint 2 and radius 0.02. (See Figure 3.) If we can keep $Q(x)$ in the second of these, we are automatically keeping it in the first, and that is what our critic demands.

Figure 3 Replacing an arbitrary neighborhood by a symmetric neighborhood

Hence let us assume that the challenge is of the form

$$N = (2 - \varepsilon, 2 + \varepsilon)$$

where ε is any positive number (the radius of N). Our problem is to name a punctured neighborhood of 1 (say M) such that

$$x \in M \Rightarrow Q(x) \in N$$

As in Example 1(b) it is just a matter of solving the appropriate inequalities. Our critic is asking us to satisfy

$$2 - \varepsilon < Q(x) < 2 + \varepsilon$$

so we write

$$\begin{aligned} 2 - \varepsilon < \frac{x^2 - 1}{x - 1} < 2 + \varepsilon &\Leftrightarrow 2 - \varepsilon < x + 1 < 2 + \varepsilon \quad (x \neq 1) \\ &\Leftrightarrow 1 - \varepsilon < x < 1 + \varepsilon \quad (x \neq 1) \end{aligned}$$

Our response should now be clear. We choose M to be the neighborhood $(1 - \varepsilon, 1 + \varepsilon)$ with 1 deleted. Then (following the above implications backward) we have

$$x \in M \Rightarrow Q(x) \in N$$

which is what our critic must believe to be satisfied.

A graph helps clarify the procedure in Example 1(c). See Figure 4, in which we show M and N on the x and y axes, respectively. Our critic names a symmetric neighborhood of 2 with radius $\varepsilon > 0$. We respond by naming a (punctured) neighborhood of 1. The arrows indicate that the function Q sends the points of M into N, which is what our critic demands.

Note that our choice of M is not unique. If we were to take M to be the neighborhood $(1 - \varepsilon/2, 1 + \varepsilon/2)$ with 1 deleted, the implication

$$x \in M \Rightarrow Q(x) \in N$$

would still be correct. The punctured neighborhood shown in Figure 4 is the largest (and simplest) we can select, but any smaller neighborhood serves as well.

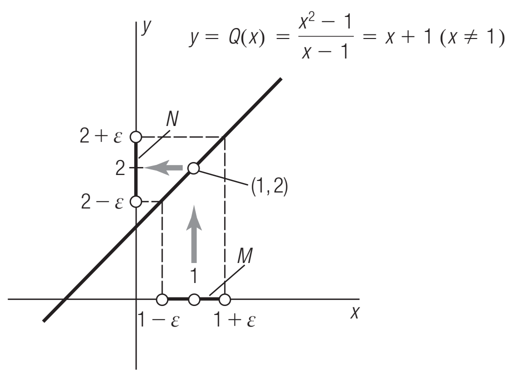


Figure 4 Sending M into N ■

■ Example 1(d)

In Example 1(a) we asked how we could argue with a critic who suggests that

$$\lim_{x\to 1} Q(x) = 2.001$$

Now we are in a position to say that the critic is definitely wrong. We know from Example 1(c) that $Q(x)$ can be made arbitrarily close to 2 by taking x sufficiently close to 1. In particular we can box $Q(x)$ away from 2.001 by forcing it into the neighborhood $N = (1.9995, 2.0005)$. (See Figure 5.) In other words the statement

$$\lim_{x\to 1} Q(x) = 2.001$$

is false. By a similar argument we can show that if L is any number except 2, the statement

$$\lim_{x\to 1} Q(x) = L$$

is false, for we can box $Q(x)$ away from L by keeping x close to 1.

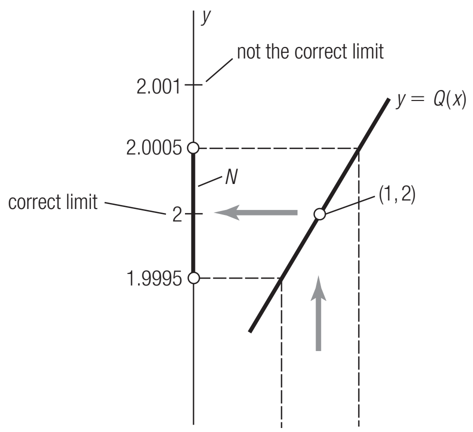


Figure 5 Boxing $Q(x)$ away from 2.001 ■

■ Example 2

Sometimes the neighborhoods involved in evaluating a limit are *one-sided.* Consider, for example, the statement

$$\lim_{x\to 1} \sqrt{x-1} = 0$$

The graph of $f(x) = \sqrt{x-1}$ is shown in Figure 6; it is the upper half of the parabola $y^2 = x - 1$. (Why?) A critic who doubts that the limit of $f(x)$ as $x \to 1$ is 0 would challenge us to confine $f(x)$ to a neighborhood of 0, say $N = (-\varepsilon, \varepsilon)$, where $\varepsilon > 0$. Since the domain of f is $\{x: x \geq 1\}$, we cannot work with an ordinary neighborhood of 1 in response. Values of x to the left of 1 cannot be used at all. What we do instead is name a *right neighborhood* of 1, as shown in Figure 6. To figure out what M should be, we follow the horizontal line from ε on the y axis until we hit the graph and then proceed down to the x axis. The point we hit is the right-hand endpoint of M, say b. To find b, we observe that it must satisfy $f(b) = \varepsilon$, that is,

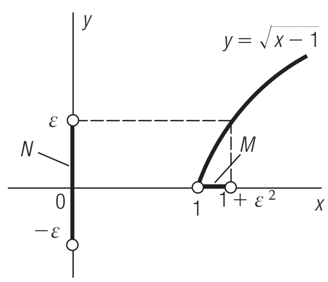


Figure 6 $\lim_{x\to 1} \sqrt{x-1} = 0$

$$\sqrt{b-1} = \varepsilon \Rightarrow b - 1 = \varepsilon^2$$
$$\Rightarrow b = 1 + \varepsilon^2$$

Hence M is the open interval $(1, 1 + \varepsilon^2)$. To confirm that it works (without depending on the picture), we write

$$\begin{aligned} x \in M &\Rightarrow 1 < x < 1 + \varepsilon^2 \\ &\Rightarrow 0 < x - 1 < \varepsilon^2 \\ &\Rightarrow 0 < \sqrt{x-1} < \varepsilon \qquad \text{(Order Property 8, Section 1.1)} \\ &\Rightarrow f(x) \in N \end{aligned}$$
■

These examples should help explain the following definition of limit.

Limit of a Real Function

Let f be a real function with domain D and suppose that a is a real number having a punctured neighborhood in D. (This insures that all points near a, with the possible exception of a itself, are points of D. In other words, $f(x)$ is defined for all x near a, but $f(a)$ may be undefined.) The statement

$$\lim_{x \to a} f(x) = L$$

where L is a real number is defined to mean the following:

> Given any neighborhood of L (say N), there is a punctured neighborhood of a (say M) such that $x \in M \Rightarrow f(x) \in N$.

If a has only a right [left] neighborhood in D, we replace "punctured neighborhood" by "right [left] neighborhood."

■ Example 3

Prove that $\lim_{x \to 2} x^2 = 4$.

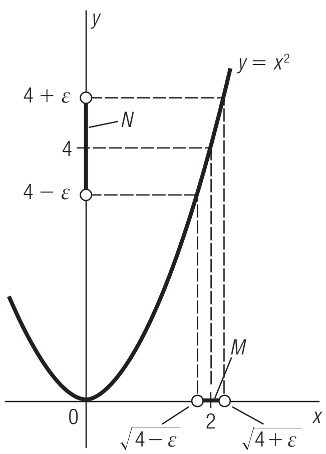


Figure 7 Naming M when N is given

Solution

We start by assuming that a critic has named a neighborhood of 4, say

$$N = (4 - \varepsilon, 4 + \varepsilon) \quad \text{where } \varepsilon > 0$$

Our problem is to name a neighborhood of 2 (say M) such that

$$x \in M \Rightarrow x^2 \in N$$

(No need to puncture it this time. Why?) From the points $4 - \varepsilon$ and $4 + \varepsilon$ on the y axis (Figure 7), follow horizontal lines to the graph of $y = x^2$ $(x \geq 0)$, then vertical lines to the x axis. The corresponding points on the x axis are $\sqrt{4 - \varepsilon}$ and $\sqrt{4 + \varepsilon}$, respectively. (Why?) They serve as endpoints of M. It is geometrically apparent that $x \in M \Rightarrow x^2 \in N$; if our critic wants algebraic confirmation, we write

$$
\begin{aligned}
x \in M &\Rightarrow \sqrt{4-\varepsilon} < x < \sqrt{4+\varepsilon} \\
&\Rightarrow 4-\varepsilon < x^2 < 4+\varepsilon \\
&\Rightarrow x^2 \in N
\end{aligned}
$$

■

Many calculus books give a definition of limit that does not mention neighborhoods. To see how this is done, assume that the neighborhood N is symmetric about L with radius $\varepsilon > 0$. Then

$$
\begin{aligned}
f(x) \in N &\Leftrightarrow L-\varepsilon < f(x) < L+\varepsilon \\
&\Leftrightarrow -\varepsilon < f(x) - L < \varepsilon \\
&\Leftrightarrow |f(x) - L| < \varepsilon
\end{aligned}
$$

Similarly, assume that M is symmetric about a with radius $\delta > 0$. Then

$$
\begin{aligned}
x \in M &\Leftrightarrow a-\delta < x < a+\delta \qquad (x \neq a) \\
&\Leftrightarrow -\delta < x - a < \delta \qquad (x \neq a) \\
&\Leftrightarrow 0 < |x-a| < \delta
\end{aligned}
$$

The statement $\lim_{x\to a} f(x) = L$ is therefore equivalent to the following:

Given $\varepsilon > 0$, there is a $\delta > 0$ such that

$$0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

If a has only a right neighborhood in the domain of f, we replace

$$0 < |x-a| < \delta \quad \text{by} \quad 0 < x-a < \delta$$

(to keep $x > a$). If there is only a left neighborhood of a in the domain, we replace

$$0 < |x-a| < \delta \quad \text{by} \quad 0 < a-x < \delta$$

(to keep $x < a$).

To see how this "ε-δ definition" works in practice, refer to our earlier examples. If $\varepsilon > 0$ is named in Example 1(c), where we proved that

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$$

the corresponding $\delta > 0$ is $\delta = \varepsilon$. That is,

$$0 < |x-1| < \varepsilon \Rightarrow \left| \frac{x^2-1}{x-1} - 2 \right| < \varepsilon$$

In Example 2, $\lim_{x\to 1} \sqrt{x-1} = 0$, we name $\delta = \varepsilon^2$ (and use a right neighborhood). That is,

$$0 < x-1 < \varepsilon^2 \Rightarrow \left| \sqrt{x-1} - 0 \right| < \varepsilon$$

In Example 3, on the other hand, the ε-δ definition is not as convenient. The neighborhood $M = (\sqrt{4-\varepsilon}, \sqrt{4+\varepsilon})$ is not symmetric about the point 2. To name $\delta > 0$ such that

$$|x-2| < \delta \Rightarrow |x^2 - 4| < \varepsilon$$

we would have to figure out which of the endpoints $\sqrt{4-\varepsilon}$ and $\sqrt{4+\varepsilon}$ is closer to 2 and cut down M to a symmetric neighborhood with the smaller distance as radius. There is not much point in taking the trouble! This illustrates the fact that we may choose between the ε-δ definition and the neighborhood definition as the situation demands.

■ Example 4

To prove that $\lim_{x\to 2}(x-2)=4$, we suppose that a critic has named $\varepsilon>0$. Our problem is to name $\delta>0$ such that

$$|x-2|<\delta \Rightarrow |(3x-2)-4|<\varepsilon$$

(Note that we do not insist on $0<|x-2|$ in this case because it is unnecessary to keep $x\neq 2$.) Since

$$\begin{aligned}|(3x-2)-4|<\varepsilon &\Leftrightarrow |3(x-2)|<\varepsilon\\ &\Leftrightarrow |(x-2)|<\frac{\varepsilon}{3}\end{aligned}$$

we name $\delta=\varepsilon/3$. Then (following the implications backward) we have

$$|x-2|<\delta \Rightarrow |x-2|<\frac{\varepsilon}{3} \Rightarrow |3(x-2)|<\varepsilon \Rightarrow |(3x-2)-4|<\varepsilon$$ ■

■ Example 5

To show that $\lim_{x\to 2}(1/x)=\frac{1}{2}$, we suppose as usual that we are confronting a skeptic, who gives us a neighborhood of $\frac{1}{2}$ with radius $\varepsilon>0$. Our problem is to force $1/x$ to lie in this neighborhood by restricting x to be near 2.

The easiest procedure is to look at the graph (Figure 8). Here we have shown the challenger's neighborhood (cut down, if necessary, to exclude 0) as an interval on the y axis with $\frac{1}{2}$ as midpoint. To keep $y=1/x$ in this neighborhood, we restrict x to the neighborhood of 2 shown on the x axis. This neighborhood is not symmetric about 2, but our first definition of limit does not require it to be.

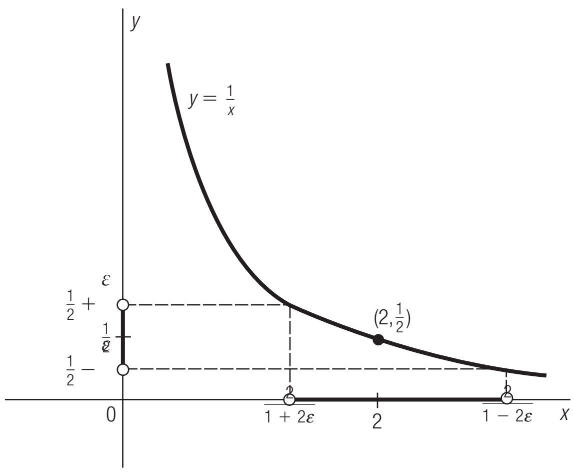


Figure 8 $\frac{1}{x}\to\frac{1}{2}$ as $x\to 2$ ■

Now we look at some limits that fail to exist.

■ Example 6

Let $f(x) = |x|/x$ and consider

$$\lim_{x\to 0} f(x)$$

As you can see from Figure 9, this limit does not exist since $f(x)$ is 1 or -1 depending on whether $x > 0$ or $x < 0$. To convince a skeptic by appealing to the definition, we would assume the contrary, namely

$$\lim_{x\to 0} f(x) = L$$

where L is a real number. No matter what L is, there is no way to confine $f(x)$ to a small neighborhood of L by keeping x close to 0 (because every neighborhood of 0 contains both positive and negative values of x, the corresponding values of f being 1 and -1). ■

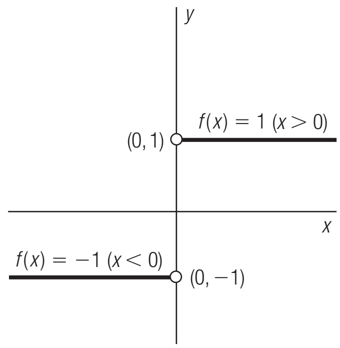


Figure 9 What does $\dfrac{|x|}{x}$ approach when $x \to 0$?

In Example 6 we may confine x to a right neighborhood of 0 and evaluate

$$\lim_{x\to 0^+} f(x) = 1$$

The notation $x \to 0^+$ means that x approaches 0 through positive values; such a limit is called a **right-handed limit**. Similarly, the **left-handed limit** in this example is

$$\lim_{x\to 0^-} f(x) = -1$$

where $x \to 0^-$ means that x approaches 0 through negative values. One way of deciding that

$$\lim_{x\to 0} f(x) \text{ does not exist}$$

is to compute the one-sided limits and observe that they are different. (It can be proved that when the domain permits approach from both sides, the ordinary limit exists if and only if the one-sided limits are equal.)

In view of these remarks, you may want to go back to Example 2, where we evaluated

$$\lim_{x\to 1} \sqrt{x-1} = 0$$

by confining x to a right neighborhood of 1. Is this an ordinary limit or a right-handed limit? Our answer is that it is both! The domain of $f(x) = \sqrt{x-1}$ does not permit x to be less than 1, so it is a matter of indifference whether we write

$$\lim_{x\to 1} \sqrt{x-1} \quad \text{or} \quad \lim_{x\to 1^+} \sqrt{x-1}$$

In Example 6, on the other hand, the domain allows $x \to 0$, $x \to 0^+$, or $x \to 0^-$, and hence it is necessary to distinguish between them.

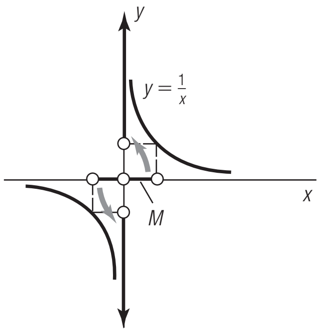


Figure 10 Infinite jump at the origin

■ Example 7

The failure of the limit to exist in Example 6 is due to a finite jump in the graph. However, there are several ways a limit may fail to exist. Consider

$$\lim_{x \to 0} f(x) \quad \text{where } f(x) = \frac{1}{x}$$

(See Figure 10.) It is clear from the picture that if M is any (punctured) neighborhood of 0, the statement $x \in M$ does not imply any statement of the type $|f(x) - L| < \varepsilon$. The most we can say is that $f(x)$ is unbounded when $x \to 0$ (increasing or decreasing depending on whether $x \to 0^+$ or $x \to 0^-$). There is no number L such that $f(x)$ is near L for all $x \in M$. ■

■ Example 8

Another way a limit may fail to exist is by oscillation. The graph of $f(x) = \sin(1/x)$ is shown in Figure 11. On any given piece of this curve there is no problem in describing the action; we are on a kind of sine wave between $y = 1$ and $y = -1$. However, there is a compression of the wave near the y axis, an increase in frequency that is boundless as $x \to 0$.

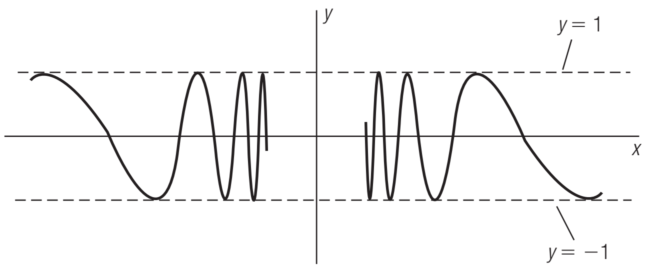


Figure 11 Graph of $y = \sin \frac{1}{x}$

Between any two points of the curve on opposite sides of the y axis there are infinitely many vibrations, a situation that is hard to visualize and impossible to draw. We conclude that

$$\lim_{x \to 0} \sin \frac{1}{x} \quad \text{does not exist.}$$ ■

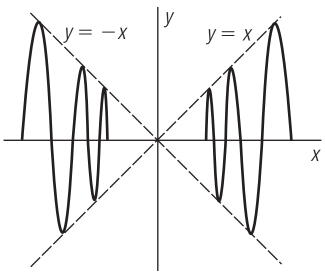


Figure 12 Graph of $y = x \sin \frac{1}{x}$

■ Example 9

The difficulty in Example 8 may be removed by "damping" the sine wave. Let $f(x) = x \sin(1/x)$, the damping factor being x. This has the same frequency of vibration as the function $y = \sin(1/x)$, so we are not curing the infinity of oscillations. But now the graph lies between the lines $y = x$ and $y = -x$, as shown in Figure 12. To see why, observe first that

$$-1 \le \sin \frac{1}{x} \le 1 \qquad \text{for all } x \ne 0 \quad \text{(Why?)}$$

Now multiply each side by x (preserving the inequalities if $x > 0$ and reversing them if $x < 0$). This yields

$$-x \le f(x) \le x \qquad \text{if } x > 0$$
$$-x \ge f(x) \ge x \qquad \text{if } x < 0$$

Hence, in any case, $f(x)$ is between x and $-x$. It follows that

$$\lim_{x\to 0} f(x) = 0$$

because $f(x)$ is boxed in between two functions ($y = x$ and $y = -x$) whose common limit as $x \to 0$ is 0. (This example illustrates the Squeeze Play Theorem, which will be stated in the next section.) ■

Problem Set 2.2

1. To verify that $\lim_{x\to 2}(3x+1) = 7$, assume that a critic has named a neighborhood of 7 of the form $N = (7-\varepsilon, 7+\varepsilon)$, where $\varepsilon > 0$. Name a neighborhood of 2 (say M) such that $x \in M \Rightarrow 3x+1 \in N$. Why is it unnecessary to puncture M?

2. To verify that $\lim_{x\to 3}(6x-13) = 5$, assume that a critic has named a neighborhood of 5 of the form $N = (5-\varepsilon, 5+\varepsilon)$, where $\varepsilon > 0$. Name a neighborhood of 3 (say M) such that $x \in M \Rightarrow 6x-13 \in N$. Why is it unnecessary to puncture M?

3. To verify that
$$\lim_{x\to 3}\frac{x^2-9}{x-3} = 6$$
let $N = (6-\varepsilon, 6+\varepsilon)$ be a given neighborhood of 6. Name a punctured neighborhood of 3 (say M) such that
$$x \in M \Rightarrow \frac{x^2-9}{x-3} \in N$$
Why must M be punctured?

4. To verify that
$$\lim_{x\to 5}\frac{x^2-25}{x-5} = 10$$
let $N = (10-\varepsilon, 10+\varepsilon)$ be a given neighborhood of 10. Name a punctured neighborhood of 5 (say M) such that
$$x \in M \Rightarrow \frac{x^2-25}{x-5} \in N$$
Why must M be punctured?

Each of the following limits is of the form $\lim_{x\to a} f(x) = L$. As in Problems 1 through 4, verify the limit by assuming as given a neighborhood of L of the form $N = (L-\varepsilon, L+\varepsilon)$ and then naming a neighborhood of a (say M) such that $x \in M \Rightarrow f(x) \in N$. Puncture M when necessary.

5. $\lim_{x\to 3}(2x+4) = 10$
6. $\lim_{x\to 3/2}\dfrac{4x^2-9}{2x-3} = 6$
7. $\lim_{x\to 4/3}\dfrac{9x^2-16}{3x-4} = 8$
8. $\lim_{x\to 4}\sqrt{x} = 2$
9. $\lim_{x\to 1}\sqrt{x} = 1$
10. $\lim_{x\to 0}\sqrt{1-x} = 1$
11. $\lim_{x\to 3} x^2 = 9$
12. $\lim_{x\to 4} x^2 = 16$
13. $\lim_{x\to 1}(4-x^2) = 3$
14. $\lim_{x\to 1}(x^2+6x) = 7$ *Hint*: Complete the square in x^2+6x.
15. $\lim_{x\to 2}(x^2-2x) = 0$
16. $\lim_{x\to 1}(x^2-4x+3) = 0$
17. $\lim_{x\to 1}\dfrac{1}{x} = 1$
18. $\lim_{x\to 2}\dfrac{6}{x} = 3$
19. $\lim_{x\to 5^+}\sqrt{x-5} = 0$ (M must be a right neighborhood of 5.)
20. $\lim_{x\to 1^-}\sqrt{1-x^2} = 0$ (M must be a left neighborhood of 1.)
21. $\lim_{x\to 2^+}\sqrt{x^2-4} = 0$ (M must be a right neighborhood of 2.)
22. To verify that $\lim_{x\to 1} 3x = 3$, let $\varepsilon > 0$ be given. Name $\delta > 0$ such that
$$|x-1| < \delta \;\Rightarrow\; |3x-3| < \varepsilon$$

23. To verify that

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = 4$$

let $\varepsilon > 0$ be given. Name $\delta > 0$ such that

$$0 < |x-2| < \delta \Rightarrow \left|\frac{x^2-4}{x-2} - 4\right| < \varepsilon$$

24. To verify that $\lim_{x\to 0} (1-x^2) = 1$, let $\varepsilon > 0$ be given. Name $\delta > 0$ such that $|x| < \delta \Rightarrow |(1-x^2)-1| < \varepsilon$.

25. To verify that $\lim_{x\to 0} \sin x = 0$, let $\varepsilon > 0$ be given. Name $\delta > 0$ such that $|x| < \delta \Rightarrow |\sin x| < \varepsilon$.

Hint: Use the contraction property of sine

(Problem 49, Section 1.6).

26. To verify that $\lim_{x\to 2^-} (1+\sqrt{4-x^2}) = 1$, let $\varepsilon > 0$ be given. Name $\delta > 0$ such that

$$0 < 2-x < \delta \Rightarrow |(1+\sqrt{4-x^2})-1| < \varepsilon$$

Why do we write $0 < 2-x < \delta$ instead of $0 < |x-2| < \delta$?

In each of the following, evaluate the given limit, then use an appropriate definition to prove that you are correct.

27. $\lim_{x\to -1} \frac{x^2-1}{x+1}$

28. $\lim_{x\to 2} x^3$

29. $\lim_{x\to 1} (x^3+8)$

30. $\lim_{x\to 0} (1-x^2)$

31. $\lim_{x\to 0} (x^2-4)$

32. $\lim_{x\to 0} \frac{(1+x)^2-1}{x}$

33. $\lim_{x\to 0} (1-\sqrt{x})$

34. $\lim_{x\to 2} (\sqrt{x-1}+2)$

35. $\lim_{x\to 3} \sqrt{9-x^2}$

36. $\lim_{x\to 1} \sqrt{1-x^2}$

37. $\lim_{x\to 1} \frac{2}{x}$

38. $\lim_{x\to 2} \frac{4}{x-1}$

39. $\lim_{x\to 1} \frac{x}{x-2}$

40. $\lim_{x\to 0} \cos x$ *Hint*: Use the fact that $|\cos t - 1| \le |t|$ for all t (Problem 49, Section 1.6).

41. $\lim_{x\to 0} x^2 \sin\frac{1}{x}$

42. Show that if $f(x) = c$ is a constant function, then

$$\lim_{x\to a} f(x) = c$$

In each of the following, evaluate the limit (or decide that it does not exist). You need not prove that your answer is correct.

43. $\lim_{x\to 0} \frac{(3+x)^2-9}{x}$

44. $\lim_{x\to 2} \frac{x^3-8}{x-2}$

45. $\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2}$

46. $\lim_{x\to 2} \frac{x^2}{x-2}$

47. $\lim_{x\to 2} \sqrt{1-x}$

48. $\lim_{x\to 0} \tan x$

49. $\lim_{x\to 0} x^0$

50. $\lim_{x\to 3} \frac{|x|}{x}$

51. $\lim_{x\to 2^+} \frac{|x-2|}{x-2}$

52. Is it correct to say that

$$\lim_{x\to 2} \sqrt{x} = 1.414?$$

Explain.

53. Explain why

$$\lim_{x\to 2} x^2 \neq 3.99$$

by naming a neighborhood of 3.99 to which $f(x) = x^2$ cannot be confined by keeping x near 2.

54. Suppose that

$$\lim_{x\to a} f(x) = L > 0$$

Explain why there is a punctured neighborhood of a (say M) such that $f(x) > 0$ for all $x \in M$.

Note: Problem 54 shows that a function with a positive limit must have positive values for all x near the point of approach. The same statement is true with "positive" replaced by "negative." (Why?)

55. Suppose that the domain of $f(x)$ permits x to approach a from either side.

(a) Show that if one-sided limits $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ exist and have a common value L, then

$$\lim_{x\to a} f(x) = L$$

(b) Conversely, if $\lim_{x\to a} f(x) = L$, why are the one-sided limits both equal to L?

56. In view of the results of Problem 55, what can you say about

$$\lim_{x\to 2} \frac{|x-2|}{x-2}?$$

57. If f is defined by the rule

$$f(x) = \begin{cases} x \text{ when } x < 1 \\ 2-x \text{ when } x > 1 \end{cases}$$

find $\lim_{x\to 1^+} f(x)$ and $\lim_{x\to 1^-} f(x)$.

What can you say about $\lim_{x\to 1} f(x)$?

2.3 Properties of Limits

The last section (a discussion of the formal definition of limit) is not essential for the evaluation of most limits encountered in the early parts of calculus. It is unnecessary to be technical when the value of a limit is apparent. Moreover, a difficult limit may often be simplified by an appropriate application of the properties of limits. Suppose, for example, that you feel like balking at the statement $\lim_{x\to 0} \tan x = 0$. Of course it is clear from the graph of tangent (Figure 7, Section 1.6) that the answer is 0, but it is not trivial to prove it. One way to avoid a confrontation with the formal definition of the last section is to observe that we *have* proved

$$\lim_{x\to 0} \sin x = 0 \qquad \text{and} \qquad \lim_{x\to 0} \cos x = 1$$

(See Problems 25 and 40 in the last section.) It seems reasonable to conclude that

$$\tan x = \frac{\sin x}{\cos x} \to \frac{0}{1} = 0 \qquad \text{as } x\to 0$$

or (more formally)

$$\lim \tan x = \lim \frac{\sin x}{\cos x} = \frac{\lim \sin x}{\lim \cos x} = \frac{0}{1} = 0$$

(We suppress $x\to 0$ in each limit to simplify the notation.) From this it is clear that we are making an assumption: How do we know that the limit of a quotient is the quotient of the limits?

Another example is

$$\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2} = \lim_{x\to 4} (\sqrt{x}+2) = 4$$

To defend the last step, we may use the definition directly. But if we have already done that in the case of

$$\lim_{x\to 4} \sqrt{x} = 2 \qquad \text{(Problem 8, Section 2.2)}$$

it hardly seems worthwhile to suffer through it again. A better approach is to write

$$\lim_{x\to 4} (\sqrt{x}+2) = \lim_{x\to 4} \sqrt{x} + \lim_{x\to 4} 2 = 2 + 2 = 4$$

Again, however, note the assumptions: The limit of a sum is the sum of the limits; the limit of a constant is the constant.

Perhaps these assumptions strike you as obvious, but some of them are not easy to establish, in general. In this section we state theorems about limits for future reference, offering only incomplete proofs (with more details in the problem set).

THEOREM 1 (Algebra of Limits)

Let f and g be real functions whose sum, difference, product, and quotient are defined, and suppose that

$$\lim_{x\to a} f(x) \qquad \text{and} \qquad \lim_{x\to a} g(x)$$

exist. Then

1. $\lim\,(f+g)(x) = \lim\, f(x) + \lim g(x)$
2. $\lim\,(f-g)(x) = \lim\, f(x) - \lim g(x)$
3. $\lim\,(fg)(x) = [\lim f(x)][\lim g(x)]$
4. $\lim\,(f/g)(x) = \dfrac{\lim f(x)}{\lim g(x)}$, provided that $\lim g(x) \neq 0$

That is, the limit of a sum (difference, product, quotient) of two functions is the sum (difference, product, quotient) of their limits.

In Theorem 1 we are assuming that the domains of f and g overlap, so that there is a common domain where they are both defined. Their sum, difference, product, and quotient are the functions $f+g$, $f-g$, fg, f/g defined by

$$(f+g)(x) = f(x) + g(x) \qquad (f-g)(x) = f(x) - g(x)$$
$$(fg)(x) = f(x)g(x) \qquad (f/g)(x) = f(x)/g(x)$$

(The quotient requires the additional assumption that the common domain of f and g contains points at which g is not zero, so that f/g has a domain.)

It is doubtful whether a straightforward proof of Theorem 1 is very enlightening, particularly since the technical details are gruesome in places. In the problem set we will outline some ingenious ways that mathematicians have devised to avoid the difficulties; we offer a proof of (1) in an optional note at the end of this section.

In many applications of Theorem 1 two special limits are needed, namely

$$\lim_{x\to a} x = a \qquad \text{and} \qquad \lim_{x\to a} c = c$$

Each of these sounds obvious when put into words:

- The function $f(x) = x$ approaches a when x approaches a.
- The constant function $f(x) = c$ approaches c when x approaches a.

We will take them to be obvious with the observation that proofs based on the technical definition of limit in the last section can also be given.

A special case of (3) in Theorem 1 is worth noting, namely

$$\lim c\, g(x) = c \lim g(x)$$

In other words a constant factor may be "moved across the limit sign." This follows from (3) by taking $f(x) = c$ and using the fact that $\lim c = c$.

■ Example 1

Use the algebra of limits to evaluate $\lim_{x \to 1} (5x - 2)$.

Solution

You should supply a reason for each step in the following:

$$\lim_{x \to 1} (5x - 2) = \lim_{x \to 1} 5x - \lim_{x \to 1} 2 = 5 \lim_{x \to 1} x - 2 = 5(1) - 2 = 3 \quad ■$$

More generally, if $f(x) = ax + b$ is any linear function and $x_0 \in \mathcal{R}$, then

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} (ax + b) = \lim_{x \to x_0} ax + \lim_{x \to x_0} b = a \lim_{x \to x_0} x + b = ax_0 + b$$

Having gone through this tedium once, you should not repeat it! For the answer is simply $f(x_0)$. In other words, if f is a linear function and $x_0 \in \mathcal{R}$,

$$\lim_{x \to x_0} f(x) = f(x_0)$$

The limit may be found by evaluating the function at the point approached.

A function with this property is called *continuous at* x_0, an idea of such importance that we shall return to it repeatedly throughout calculus.

Continuity of a Real Function

Suppose that x_0 is a point of the domain of the real function f. We say that f is *continuous* at x_0 if

$$\lim_{x \to x_0} f(x) = f(x_0)$$

The geometrical significance of continuity at a point is that a traveler on the graph of $y = f(x)$ encounters the point $(x_0, f(x_0))$ where it "should" be. (See Figure 1.) More precisely, three conditions are met when f is continuous at x_0:

1. The graph has a point corresponding to $x = x_0$; that is, x_0 is in the domain of f.
2. A traveler on the graph can identify a point toward which he is heading as $x \to x_0$; that is, $\lim_{x \to x_0} f(x)$ exists.
3. The point in (2) that "should" be on the graph coincides with the point in (1) that *is* on the graph; that is,

$$\lim_{x \to x_0} f(x) = f(x_0)$$

In other words, the traveler encounters neither a missing point nor a displaced point at x_0, but the point he expected.

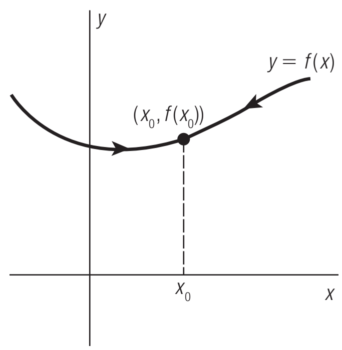


Figure 1 Continuity of f at x_0

The analytic meaning of continuity at x_0 is that the limit of $f(x)$ as $x \to x_0$ may be found by substitution of x_0 into f. We will return to this idea in Chapter 4.

Meanwhile you may assume that virtually every real function we have named in this book is continuous at each point of its domain. This includes the functions classified in Section 1.5 (constant, linear, power, root, polynomial, rational, and algebraic) and the six trigonometric functions (sine, cosine, tangent, cotangent, secant, and cosecant).

On the other hand, you should not use the formula

$$\lim_{x\to x_0} f(x) = f(x_0)$$

indiscriminately. If

$$f(x) = \frac{x^2 - 1}{x - 1}$$

for example, we cannot find $\lim_{x\to 1} f(x)$ by evaluating $f(1)$ because 1 is not in the domain. Even when the point of approach is in the domain, the formula is not necessarily correct, as the next example shows.

■ Example 2

Define the function f by

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2 \end{cases}$$

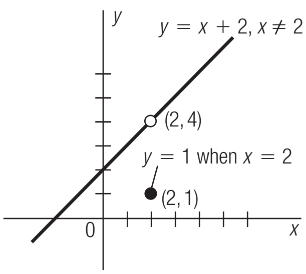


Figure 2 Straight line with a point displaced

Since $f(x) = x + 2$ when $x \neq 2$, while $f(2) = 1$, the graph of f is a straight line with a point displaced (Figure 2). As you can see,

$$\lim_{x\to 2} f(x) = 4 \neq f(2)$$

so f is not continuous at 2. ■

■ Example 3

Confirm that the function

$$f(x) = \frac{3x^2 + 1}{x - 2}$$

is continuous at each point $x_0 \neq 2$.

Solution

The problem is to show that

$$\lim_{x\to x_0} f(x) = f(x_0)$$

Suppressing $x \to x_0$ in each limit to save writing, we have

$$\lim f(x) = \lim \frac{3x^2 + 1}{x - 2} = \frac{\lim (3x^2 + 1)}{\lim (x - 2)} = \frac{\lim 3x^2 + \lim 1}{\lim x - \lim 2} = \frac{3 \lim x^2 + 1}{x_0 - 2}$$

$$= \frac{3(\lim x)(\lim x) + 1}{x_0 - 2} = \frac{3(x_0)(x_0) + 1}{x_0 - 2} = \frac{3x_0{}^2 + 1}{x_0 - 2} = f(x_0)$$

As you can see, establishing continuity of a rational function is just a matter of applying the algebra of limits repeatedly until we reduce the problem to the evaluation of "obvious" limits. ■

■ Example 4

The function tan x cot x has the constant value 1 wherever it is defined. (See Figure 3.) Its graph is the horizontal line $y = 1$ with holes punched out at $(0,1)$, $(\pm\pi/2, 1)$, $(\pm\pi, 1)$, . . . , from which it is apparent that

$$\lim_{x\to 0} \tan x \cot x = 1$$

On the other hand, suppose we apply (3) of Theorem 1, writing

$$\lim_{x\to 0} \tan x \cot x = \left(\lim_{x\to 0} \tan x\right)\left(\lim_{x\to 0} \cot x\right) = (0)\left(\lim_{x\to 0} \cot x\right) = 0$$

If this is correct, we have proved that $1 = 0$. Before reading on, can you find the fallacy?

The point is that $\lim f(x)g(x) = [\lim f(x)][\lim g(x)]$ only when the limits on the right side exist. It is tempting to argue that $(0)(\lim \cot x) = 0$ on the premise that "zero times anything is zero." The correct statement, of course, is that zero times any real number is zero. Since

$$\lim_{x\to 0} \cot x \text{ does not exist} \qquad \text{(Figure 8, Section 1.6)}$$

our "equation" reads $1 = (0)$(horseradish). ■

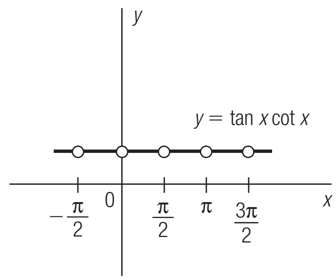


Figure 3 Graph of $y = \tan x \cot x$

The next theorem is easier to illustrate than it is to state. Hence we present some examples first.

■ Example 5

To find $\lim_{x\to 0} \sqrt{\cos x}$, we reason intuitively as follows. As $x \to 0$, cos x approaches 1. The square root of a number close to 1 is itself close to 1. Hence

$$\lim_{x\to 0} \sqrt{\cos x} = 1$$

An analysis of this reasoning reveals that we are treating $\sqrt{\cos x}$ as a *composite function* (cosine followed by square root). The "inside function" is $g(x) = \cos x$; the "outside function" is $f(x) = \sqrt{x}$; the composition is

$$f[g(x)] = f(\cos x) = \sqrt{\cos x}$$

What we did to evaluate the limit was to work from the inside out. That is, we ignored the outside function temporarily while we determined that $\cos x \to 1$ as $x \to 0$. Then we applied the outside function to obtain $\sqrt{\cos x} \to \sqrt{1}$ as $x \to 0$. In symbols,

$$\lim \sqrt{\cos x} = \sqrt{\lim \cos x} = \sqrt{1} = 1$$

or $\lim f[g(x)] = f[\lim g(x)]$.

The heart of the matter is the formula

$$\lim f[g(x)] = f[\lim g(x)]$$

which indicates how the limit of a composition is found by working from the inside out. An informal way of putting it is that the symbols "lim" and "f" are interchanged. Instead of finding the limit of f we evaluate f of the limit. (Sufficient conditions for this to work are discussed after Example 6.) ■

■ Example 6

To find $\lim_{x\to 1} (3x - 1)^3$, we could apply (3) of Theorem 1 repeatedly, writing

$$\lim (3x - 1)^3 = [\lim (3x - 1)][\lim (3x - 1)][\lim (3x - 1)] = 2^3 = 8$$

It is easier, however, to think of $(3x - 1)^3$ as a composition, namely the linear function $g(x) = 3x - 1$ followed by the power function $f(x) = x^3$. Then

$$\lim (3x - 1)^3 = \lim f[g(x)] = f[\lim g(x)] = f(2) = 8$$

When you get used to this idea, you will not need to introduce the functional symbols f and g. Just interchange "lim" and "outside function," writing

$$\lim (3x - 1)^3 = [\lim (3x - 1)]^3 = 2^3 = 8$$

In other words, work from the inside out. ■

The obvious question to raise at this point is under what conditions does the formula

$$\lim_{x\to a} f[g(x)] = f\left[\lim_{x\to a} g(x)\right]$$

apply? The obvious answer is that $\lim g(x)$ must exist and lie in the domain of f, since otherwise the right side would not make sense. It is not quite that simple, however. The interchange of "lim" and "f" requires f to have the property

$$\lim_{u\to u_0} f(u) = f(u_0) \quad \text{where } u_0 = \lim_{x\to a} g(x)$$

(That is, f should be continuous at u_0.) This condition is equivalent to

$$\lim_{u\to u_0} f(u) = f\left(\lim_{u\to u_0} u\right) \qquad \text{(why?)}$$

which is precisely the interchange property needed.

THEOREM 2 (Composite Function Theorem)

Let f and g be real functions whose composition $f[g(x)]$ is defined. Then

$$\lim_{x\to a} f[g(x)] = f\left[\lim_{x\to a} g(x)\right]$$

provided that $u_0 = \lim_{x\to a} g(x)$ exists and f is continuous at u_0.

We offer an informal proof of Theorem 2 in an optional note at the end of this section. Also see Problems 43 and 44 in the problem set.

■ **Example 7**

Explain why the Composite Function Theorem does not apply to

$$\lim_{x\to\pi} \tan \frac{x}{2}$$

Solution

It is incorrect to write

$$\lim_{x\to\pi} \tan \frac{x}{2} = \tan\left(\lim_{x\to\pi} \frac{x}{2}\right) = \tan \frac{\pi}{2}$$

For although the limit of the inside function exists, namely

$$u_0 = \lim_{x\to\pi} \frac{x}{2} = \frac{\pi}{2}$$

the outside function (tangent) is not continuous at u_0. [In fact, $\tan(\pi/2)$ is not even defined.]

Despite this remark, it is still possible to use the idea of the Composite Function Theorem. When x approaches π, the inside function $x/2$ approaches $\pi/2$. Since tangent is unbounded when its input is allowed to approach $\pi/2$ (Figure 7, Section 1.6), we conclude that

$$\lim_{x\to\pi} \tan \frac{x}{2} \text{ does not exist}$$ ■

Our next theorem is so geometrically apparent (Figure 4) that we omit its proof altogether. (See the problem set, however, for hints on how to construct one.) This statement is called the Sandwich Theorem by some. Another descriptive title (borrowed from baseball) is the *Squeeze Play Theorem.*

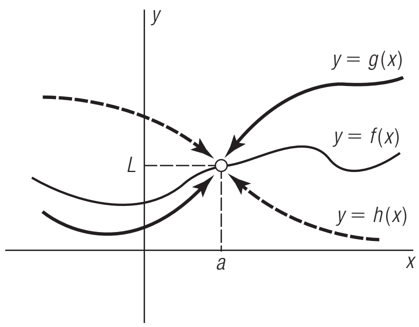


Figure 4 Squeeze Play Theorem

THEOREM 3 (Squeeze Play Theorem)

Suppose that $f(x)$ is between $g(x)$ and $h(x)$ for all x near a. If $g(x)$ and $h(x)$ have a common limit as $x \to a$ (say L), then

$$\lim_{x\to a} f(x) = L$$

An application of the Squeeze Play Theorem may be seen in Example 9, Section 2.2, where we evaluated

$$\lim_{x\to 0} x \sin \frac{1}{x}$$

Since $f(x) = x \sin(1/x)$ is between $g(x) = x$ and $h(x) = -x$ for all $x \neq 0$ (Figure 12, Section 2.2) and since

$$\lim_{x\to 0} g(x) = \lim_{x\to 0} h(x) = 0$$

the limit of $f(x)$ must be 0.

Optional Note ***(on the proof of (1) in Theorem 1)***

Let

$$A = \lim_{x\to a} f(x) \qquad \text{and} \qquad B = \lim_{x\to a} g(x)$$

The problem is to prove that

$$\lim_{x\to a} (f+g)(x) = A + B$$

In other words, given $\varepsilon > 0$, we must name $\delta > 0$ such that

$$0 < |x-a| < \delta \Rightarrow |(f+g)(x) - (A+B)| < \varepsilon$$

But

$$|(f+g)(x) - (A+B)| = |[f(x) - A] + [g(x) - B]| \leq |f(x) - A| + |g(x) - B|$$

(See the Triangle Inequality in Section 1.1.) To force this to be less than ε, we need only force $f(x)$ and $g(x)$ to be within $\varepsilon/2$ units of A and B, respectively. Since we know from the definitions of A and B that this can be done by keeping x sufficiently close to a, the proof should be clear in outline.

To be precise in detail, we know there are positive numbers δ_1 and δ_2 such that

$$0 < |x-a| < \delta_1 \Rightarrow |f(x) - A| < \frac{\varepsilon}{2}$$

and

$$0 < |x-a| < \delta_2 \Rightarrow |g(x) - B| < \frac{\varepsilon}{2}$$

Let δ be the smaller of δ_1 and δ_2. Then

$$0 < |x-a| < \delta \Rightarrow |f(x) - A| < \frac{\varepsilon}{2} \text{ and } |g(x) - B| < \frac{\varepsilon}{2}$$

$$\Rightarrow |(f+g)(x) - (A+B)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Optional Note ***(on the proof of Theorem 2)***

Instead of a formal proof, we offer the following intuitive argument. To show that

$$\lim_{x\to a} f[g(x)] = f(u_0)$$

we suppose that a skeptic has named a neighborhood of $f(u_0)$. (See Figure 5.)

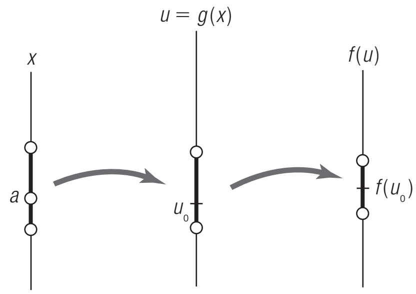


Figure 5 Intuitive argument for composite function theorem

Our problem is to confine $f[g(x)]$ to this neighborhood by keeping x sufficiently close to a. We do this in a chain of steps:

1. Confine $f[g(x)]$ to the given neighborhood of $f(u_0)$ by controlling $u = g(x)$. We know this can be done because

$$\lim_{u \to u_0} f(u) = f(u_0)$$

2. Force $g(x)$ as close to u_0 as required in (1) by keeping x near a. We know this can be done because

$$\lim_{x \to a} g(x) = u_0$$

A formal version of this argument is outlined in Problems 43 and 44 in the problem set.

Problem Set 2.3

Evaluate each of the following by using properties of limits.

1. $\lim_{x \to 1} (7x + 3)$
2. $\lim_{x \to 2} (9x - 5)$
3. $\lim_{x \to 1} \dfrac{x^2 - 1}{x - 1}$
4. $\lim_{x \to 2} (5 - x^2)$
5. $\lim_{x \to 2} (x^3 + x)$
6. $\lim_{x \to 5} \sqrt{x - 5}$ *Hint*: Use Theorem 2 with continuity of root functions.
7. $\lim_{x \to 4} \sqrt{x(x - 3)}$
8. $\lim_{x \to 1} \sqrt{9 - x^2}$
9. $\lim_{x \to 4} x^{3/2}$
10. $\lim_{x \to 8} x^{2/3}$
11. $\lim_{x \to 0} (16 - x)^{3/4}$
12. $\lim_{x \to 1} \dfrac{1}{x}$
13. $\lim_{x \to 2} \dfrac{4}{x}$
14. $\lim_{x \to 0} \dfrac{x - 3}{2x + 1}$
15. $\lim_{x \to 1} \dfrac{3x + 1}{3x - 1}$
16. $\lim_{x \to 3} \dfrac{x - 3}{3 - x}$
17. $\lim_{x \to -1} \dfrac{1 - 2x}{x^2 - x + 4}$
18. $\lim_{x \to 0} \dfrac{x^3 + 1}{x + 1}$
19. $\lim_{x \to 1} \dfrac{x^3 - 1}{x - 1}$
20. $\lim_{x \to 2} \dfrac{x^3 - 8}{x - 2}$
21. $\lim_{x \to 3} \dfrac{x^2 + x - 12}{x^2 - 7x + 12}$
22. $\lim_{x \to 0} \dfrac{\sqrt{x + 4} - 2}{x}$ *Hint*: Rationalize the numerator.
23. $\lim_{x \to 2} \dfrac{x - 2}{\sqrt{4 - x^2}}$
24. $\lim_{x \to 2} \left(\dfrac{1}{x - 2} - \dfrac{4}{x^2 - 4} \right)$
25. $\lim_{x \to 0} \sin 2x$ *Hint*: Use Theorem 2 with continuity of trigonometric functions.
26. $\lim_{x \to 0} \cos 2x$
27. $\lim_{x \to 0} \sec \sqrt{x}$
28. $\lim_{x \to 0} \dfrac{x}{\cos x}$
29. $\lim_{x \to 0} x \cos \dfrac{1}{x}$ *Hint*: Use Theorem 3.
30. $\lim_{x \to \pi} \sqrt{1 - \sin^2 x}$
31. $\lim_{x \to 3\pi/2} \sqrt{1 - \cos^2 x}$
32. $\lim_{x \to \pi/4} |\sec x + \tan x|$

In the text we said that virtually every function named in this book is continuous at every point of its domain. The following problems are designed to justify that statement (in part).

33. A polynomial is a function of the form

$$P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

where n is a nonnegative integer and each a_k is a real number. Show that if $x_0 \in \mathcal{R}$, then

$$\lim_{x\to x_0} P(x) = P(x_0)$$

(Thus a polynomial is continuous at every point $x_0 \in \mathcal{R}$.) *Hint*: We already know that

$$\lim_{x\to x_0} c = c \qquad \text{and} \qquad \lim_{x\to x_0} x = x_0$$

34. We know that sine and cosine are continuous at 0, that is,

$$\lim_{x\to 0} \sin x = \sin 0 \qquad \text{and} \qquad \lim_{x\to 0} \cos x = \cos 0$$

(See Problems 25 and 40 , Section 2.2.) To prove that sine is continuous at any $x_0 \in \mathcal{R}$, we lift ourselves up by our bootstraps:

(a) Confirm that

$$\begin{aligned}\sin x &= \sin[(x - x_0) + x_0] \\ &= \sin(x - x_0)\cos x_0 + \cos(x - x_0)\sin x_0\end{aligned}$$

(b) Why does it follow that

$$\begin{aligned}\lim_{x\to x_0} \sin x = \cos x_0 \cdot \lim_{x\to x_0} \sin(x - x_0) \\ + \sin x_0 \cdot \lim_{x\to x_0} \cos(x - x_0)?\end{aligned}$$

(c) Now explain why $\lim_{x\to x_0} \sin x = \sin x_0$.

35. Prove that cosine is continuous at every $x_0 \in \mathcal{R}$ by imitating Problem 34.

36. Why does it follow from Problems 34 and 35 that tangent, cotangent, secant, and cosecant are continuous wherever they are defined?

In the text we said that the technical details of the proof of Theorem 1 are gruesome in places. There are clever ways to avoid them, however; if you are interested, do Problems 37–42.

37. Now prove (2) of Theorem 1.

38. Use the definition of limit to show that if

$$\lim_{x\to a} f(x) = 0 \qquad \text{and} \qquad \lim_{x\to a} g(x) = 0$$

then $\lim_{x\to a} (fg)(x) = 0$. (This proves (3) of Theorem 1 in the special case where the limits of f and g are 0.)

39. Now prove (3) of Theorem 1 by letting

$$A = \lim_{x\to a} f(x) \qquad \text{and} \qquad B = \lim_{x\to a} g(x)$$

and by writing

$$(fg)(x) = [f(x) - A][g(x) - B] + Bf(x) + Ag(x) - AB$$

40. Use the definition of limit to show that if $u_0 \neq 0$, then

$$\lim_{u\to u_0} \frac{1}{u} = \frac{1}{u_0}$$

Hint: Imitate Example 5, Section 2.2.

41. Suppose that $\lim_{x\to a} g(x)$ exists and is not 0. Why does it follow from Problem 40 that

$$\lim_{x\to a} \frac{1}{g(x)} = \frac{1}{\lim_{x\to a} g(x)}?$$

Hint: Use the Composite Function Theorem. (*Objection*: That comes after Theorem 1. *Answer*: Its proof is independent of Theorem 1. See Problems 43 and 44.)

42. Now prove (4) of Theorem 1.

43. Explain why the Composite Function Theorem in the text is equivalent to the statement that if

$$\lim_{x\to a} g(x) = u_0 \qquad \text{and} \qquad \lim_{u\to u_0} f(u) = f(u_0)$$

then $\lim_{x\to a} f[g(x)] = f(u_0)$.

44. To prove the version of the Composite Function Theorem stated in Problem 43, let $L = f(u_0)$ and assume that $\varepsilon > 0$ has been given. The problem is to name $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |f[g(x)] - L| < \varepsilon$$

(a) Explain why there is a $\rho > 0$ such that

$$|u - u_0| < \rho \Rightarrow |f(u) - L| < \varepsilon$$

(The symbol ρ is the lowercase Greek letter *rho*.)

(b) Explain why there is a $\delta > 0$ such that

$$0 < |x - a| < \delta \Rightarrow |g(x) - u_0| < \rho$$

(c) Why does it follow from parts (a) and (b) that

$$0 < |x - a| < \delta \Rightarrow |f[g(x)] - L| < \varepsilon?$$

45. To prove the Squeeze Play Theorem, we suppose that a skeptic has named a neighborhood of L (say N). The problem is to force $f(x) \in N$ by keeping x near a.

(a) Explain why $g(x)$ and $h(x)$ can be forced into N by keeping x near a.

(b) Why does it follow that (for these values of x) $f(x)$ is in N?

(c) Can you make this argument more formal, so that a confirmed skeptic would believe it?

46. Use the *table* feature of your graphing calculator to find

$$\lim_{x \to -2} 2^{(x^2 - 4)/(x + 2)}$$

(Be sure to approach 2 from both sides by selecting values of x such as 1.9, 1.99, 1.999, and 2.1, 2.01, and 2.001.)

47. Refer back to the *greatest integer function* defined in Problem 7 of Section 1.4. This function is often represented as $f(x) = [x]$. Find

$$\lim_{x \to 5^+} f(x) \quad \text{and} \quad \lim_{x \to 5^-} f(x)$$

by sketching a graph. (Do you think the left and right limits would differ for any other values of x? What sort of values?)

Find the integer function in your graphing calculator and use the *table* feature to verify your results numerically.

2.4 The Derivative of a Real Function

In Section 2.1 we discussed the problem of tangents and the problem of velocity. Let us summarize those discussions and make some general observations.

1. *Slope of a Curve.* If (x_0, y_0) is a point of the graph of the real function $y = f(x)$, we find the slope at (x_0, y_0) as follows.
   (a) Let (x, y) be a point of the graph "near" (x_0, y_0). The slope of the line containing (x_0, y_0) and (x, y) is given by the *difference quotient*

   $$Q(x) = \frac{y - y_0}{x - x_0} = \frac{f(x) - f(x_0)}{x - x_0} \qquad (x \neq x_0)$$

   (b) Bring (x, y) closer to (x_0, y_0), thereby making x approach x_0. If this causes $Q(x)$ to converge to a definite value m, we write

   $$m = \lim_{x \to x_0} Q(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

   and call m the *slope of the curve* at (x_0, y_0). Since the value of the limit depends on x_0, the slope is a new function, $m(x_0) =$ slope at (x_0, y_0), or simply $m(x) =$ slope at (x, y) if we drop the subscript.

2. *Velocity of a Moving Object.* Let $s = p(t)$ be the position at time t of an object moving in a straight line (where p is a real function). We find the velocity at time t_0 as follows.
   (a) Let t be a slightly different time. The average velocity during the time interval with endpoints t_0 and t is

   $$Q(t) = \frac{\text{change in position}}{\text{change in time}} = \frac{p(t) - p(t_0)}{t - t_0} \qquad (t \neq t_0)$$

   (b) Let t approach t_0. If this causes $Q(t)$ to converge to a definite value v, we write

   $$v = \lim_{t \to t_0} Q(t) = \lim_{t \to t_0} \frac{p(t) - p(t_0)}{t - t_0}$$

and call v the *velocity* at time t_0. Since the answer depends on t_0, the velocity is a new function, $v(t_0)$ = velocity at time t_0, or simply $v(t)$ = velocity at time t if we drop the subscript.

It is a principle of human thought and language that when we encounter distinct problems with the same answer, we ought to ignore the inessential details of the problems and invent a word for the idea they have in common. The forms of the answers in (1) and (2) are identical. So we forget about geometry (slope of a curve) and physics (velocity of a moving object) and concentrate on the mathematical substance of the process illustrated. In each case a new function is derived from a given one by evaluating the limit of a difference quotient. The new function is called the **derivative** of the given function, as in the following definition.

> Let f be a real function with domain D and suppose that x_0 is an interior point of D. The **difference quotient** associated with f at x_0 is
>
> $$Q(x) = \frac{f(x) - f(x_0)}{x - x_0} \qquad (x \neq x_0 \text{ and } x \in D)$$
>
> The **derivative of f at x_0** is the number
>
> $$f'(x_0) = \lim_{x \to x_0} Q(x)$$
>
> (provided this limit exists, in which case we call f **differentiable** at x_0).
>
> If S is a subset of the domain and f is differentiable at each point of S, we say that f is *differentiable in S*.
>
> The function f' whose domain is the set of points at which f is differentiable and whose value at x is $f'(x)$ is called the *derivative* of f.

Remark

The above definition requires x_0 to be an "interior point" of D, that is, a point with the property that neighboring points are also in D. More precisely, there is an open interval containing x_0 (a *neighborhood* of x_0) that lies in D. This guarantees that $f(x)$ is defined for all x near x_0, which is essential for evaluation of the limit of $Q(x)$ as x approaches x_0. The requirement can be relaxed in some circumstances, however. If x_0 is an endpoint of an interval in D, the limit is *one-sided*; sometimes we can still make sense of it. (See Example 3.)

■ Example 1

Find the derivative of $f(x) = x^3$.

Solution

The difference quotient associated with f at x_0 is

$$Q(x) = \frac{f(x) - f(x_0)}{x - x_0} = \frac{x^3 - x_0^3}{x - x_0} = \frac{(x - x_0)(x^2 + xx_0 + x_0^2)}{x - x_0}$$

$$= x^2 + xx_0 + x_0^2 \qquad (x \neq x_0)$$

Since

$$\lim_{x \to x_0} Q(x) = \lim_{x \to x_0} (x^2 + xx_0 + x_0{}^2) = 3x_0{}^2$$

the derivative at x_0 is $f'(x_0) = 3x_0{}^2$. Having arrived at this, we may observe that in the beginning x_0 was any real number. The subscript is superfluous; we might as well say that f' is defined by $f'(x) = 3x^2$. ■

The observation at the end of Example 1 sometimes confuses people. What you should note is that the letter used is immaterial. We may write $f'(t) = 3t^2$ or $f'(\alpha) = 3\alpha^2$ or $f'(\square) = 3\square^2$. The *form* is what counts in the definition of a function, not the symbol used to identify the independent variable. The reason for the subscript in the first place is that in the process of examining the difference quotient we needed a label for *its* independent variable. We chose x, which means that we needed a different label for the point at which the derivative is to be evaluated.

To clarify this matter further, suppose we feel like calling the original point x. Then we need another letter for the independent variable in Q; suppose we adopt z. The difference quotient associated with f at x is then

$$\begin{aligned} Q(z) &= \frac{f(z) - f(x)}{z - x} = \frac{z^3 - x^3}{z - x} = \frac{(z - x)(z^2 + zx + x^2)}{z - x} \\ &= z^2 + zx + x^2 \qquad (z \neq x) \end{aligned}$$

Since

$$\lim_{z \to x} Q(z) = \lim_{z \to x} (z^2 + zx + x^2) = 3x^2$$

the derivative at x is $f'(x) = 3x^2$.

■ Example 2

Find the derivative of $f(x) = \sqrt{x}$.

Solution

The difference quotient associated with f at x (where $x > 0$) is

$$\begin{aligned} Q(z) &= \frac{f(z) - f(x)}{z - x} = \frac{\sqrt{z} - \sqrt{x}}{z - x} = \frac{\sqrt{z} - \sqrt{x}}{(\sqrt{z} - \sqrt{x})(\sqrt{z} + \sqrt{x})} \\ &= \frac{1}{\sqrt{z} + \sqrt{x}} \qquad (z \neq x \text{ and } z \geq 0) \end{aligned}$$

The derivative at x is

$$f'(x) = \lim_{z \to x} Q(z) = \lim_{z \to x} \frac{1}{\sqrt{z} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Since x was any positive number, we conclude that f is differentiable in the interval $(0, \infty)$. ■

Note in Example 2 that the domain of f is $[0, \infty)$. We did not consider the derivative at $x = 0$ because 0 is not an interior point of the domain. Suppose we stretch the definition, however, and try it. The difference quotient associated with f at 0 is

$$Q(z) = \frac{f(z) - f(0)}{z - 0} = \frac{\sqrt{z}}{z} = \frac{1}{\sqrt{z}} \qquad (z > 0)$$

Since $Q(z)$ increases without bound as $z \to 0$,

$$\lim_{z \to 0} Q(z) \text{ does not exist}$$

and we conclude that f is not differentiable at 0. (See Example 4, Section 2.1, where we observed that the tangent at the origin is vertical and the slope is undefined.)

Despite the failure of this attempt to broaden the definition, there are times when it makes sense. See Example 3.

■ Example 3

The domain of $f(x) = x^{3/2}$ is $[0, \infty)$. Even though 0 is not an interior point, let's include it in our analysis and see what happens. The difference quotient associated with f at x (where $x \geq 0$) is

$$Q(z) = \frac{f(z) - f(x)}{z - x} = \frac{z^{3/2} - x^{3/2}}{z - x} \qquad (z \neq x \text{ and } z \geq 0)$$

To simplify this, let $a = x^{1/2}$ and $b = z^{1/2}$. Then

$$\begin{aligned} Q(z) &= \frac{b^3 - a^3}{b^2 - a^2} \\ &= \frac{(b - a)(b^2 + ba + a^2)}{(b - a)(b + a)} \\ &= \frac{b^2 + ba + a^2}{b + a} = \frac{z + \sqrt{zx} + x}{\sqrt{z} + \sqrt{x}} \end{aligned}$$

Now we have to distinguish cases. If $x > 0$, we can write

$$\begin{aligned} f'(x) &= \lim_{z \to x} Q(z) = \lim_{z \to x} \frac{z + \sqrt{zx} + x}{\sqrt{z} + \sqrt{x}} \\ &= \frac{x + \sqrt{x^2} + x}{\sqrt{x} + \sqrt{x}} \\ &= \frac{3x}{2\sqrt{x}} \qquad (\sqrt{x^2} = x \text{ because } x > 0) \\ &= \frac{3}{2}\sqrt{x} \end{aligned}$$

This argument breaks down if $x = 0$ (because $3x/(2\sqrt{x})$ is meaningless). At $x = 0$, however, the difference quotient reduces to

$$Q(z) = \frac{z^{3/2} - 0}{z - 0} = \sqrt{z} \qquad (z > 0)$$

and hence

$$f'(0) = \lim_{z\to 0} Q(z) = 0$$

According to our definition of derivative, the endpoint $x = 0$ should not have been included in this analysis. Let's agree to examine endpoints, however, and to accept the results when the limit of the difference quotient exists. Then we may summarize Example 3 by saying that $f(x) = x^{3/2}$ is differentiable in $[0,\infty)$ and

$$f'(x) = \begin{cases} \frac{3}{2}\sqrt{x} & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$$

As you can see, it is unnecessary to separate the cases $x > 0$ and $x = 0$ in the end. The formula $f'(x) = \frac{3}{2}\sqrt{x}$ (if $x \geq 0$) covers both possibilities. The graph of $y = x^{3/2}$ (unlike that of $y = x^{1/2}$, which has a vertical tangent at the origin) should be drawn with slope 0 at the origin. (See Figure 1.) ■

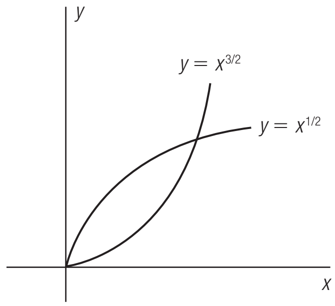


Figure 1 Horizontal and vertical tangents at the origin

The effect of our agreement in Example 3 is that we are extending the definition of derivative to include endpoints. The limit of the difference quotient in such a case is "one-sided," in the sense that

$$Q(z) = \frac{f(z) - f(x)}{z - x} \qquad \text{(where } x \text{ is an endpoint)}$$

is examined only for values of z on one side of x. In Example 3, for instance, where we wrote

$$f'(0) = \lim_{z\to 0} \frac{z^{3/2} - 0}{z - 0} = \lim_{z\to 0} \sqrt{z} = 0$$

the limit is evaluated by making z approach 0 "through positive values." (Otherwise $\sqrt{z}$ would be meaningless.)

Sometimes this sort of thing complicates the theory. (We will point it out when it does.) But on the whole, it is helpful to allow it. Would it not offend your intuition to say that $f(x) = x^{3/2}$ is differentiable only for $x > 0$? That would prevent the natural extension of the formula $f'(x) = \frac{3}{2}\sqrt{x}$ to the point $x = 0$, when it is plain from the graph that there is a horizontal tangent at the origin.

We have defined the difference quotient associated with f at x to be

$$Q(z) = \frac{f(z) - f(x)}{z - x} \qquad (z \neq x)$$

(See Figure 2.) Sometimes it is more convenient to denote the point z by the label $x + h$ ($h \neq 0$), the idea being that when h is small, $x + h$ is near x. (See Figure 3.) Then the difference quotient is a function of h, namely

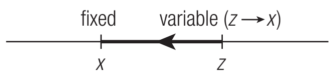


Figure 2

$$q(h) = \frac{f(x + h) - f(x)}{(x + h) - x} = \frac{f(x + h) - f(x)}{h} \qquad (h \neq 0)$$

and the derivative of f at x is

$$f'(x) = \lim_{h\to 0} q(h) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$

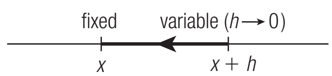


Figure 3

■ Example 4

Discuss the problem of finding the derivative of $f(x) = \sin x$.

Solution

The difference quotient associated with f at x is

$$Q(z) = \frac{\sin z - \sin x}{z - x} \qquad (z \neq x)$$

which does not simplify in any obvious way. If we substitute $x + h$ for z, however, the difference quotient becomes

$$q(h) = \frac{\sin(x + h) - \sin x}{h} \qquad (h \neq 0)$$

and a possible simplification suggests itself. Using the addition formula

$$\sin(u + v) = \sin u \cos v + \cos u \sin v$$

we have

$$q(h) = \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} = \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)$$

It is not obvious how the expressions $(\cos h - 1)/h$ and $(\sin h)/h$ behave when h approaches zero, but for the sake of argument suppose that their limits are a and b, respectively. Then

$$f'(x) = \lim_{h \to 0} q(h) = (\sin x)(a) + (\cos x)(b)$$

and the problem is solved. If you have an electronic calculator, you can probably guess what a and b are by computing the above expressions for values of h near 0. For the present, however, we will leave the problem at this stage. ■

According to the terminology we have now adopted, the two equivalent problems of Section 2.1 are solved by finding a derivative:

1. Given a curve with equation $y = f(x)$, the slope at (x, y) is the derivative $m(x) = f'(x)$.
2. Given a position function $s = p(t)$, the velocity at time t is the derivative $v(t) = p'(t)$.

These are *two* interpretations of the derivative. There are many others, as we shall see.

Calculus is a subject with a rich history. It should not surprise you that there are many alternate symbols for the derivative; you should learn to live with the ones in common use. Chief among these, besides the notation $f'(x)$ already introduced for the derivative of $y = f(x)$, are

- y' (read "y prime")
- $\dfrac{dy}{dx}$ (read "dee y, dee x"—not "dy over dx")
- $D_x y$ (read "derivative of y with respect to x")

The notation dy/dx is due to Leibniz, who regarded dy and dx as separate entities and dy/dx as a true fraction. Eventually we'll do the same; the symbol is very useful. For the present, however, you should not think of dy/dx as a fraction; it is just another notation for the derivative.

Depending on the context, the above symbols are of varying precision. They must be used with due attention to the message intended.

GRAPHING CALCULATOR CONCEPTS

Evaluating Derivatives

Your graphing calculator can evaluate the first derivative at a given value of the input variable! Investigate how online or in your graphing calculator manual. You will probably be asked to enter the function, just as you would to graph it. Then you will use the first derivative option and choose a value of x. Try to remember to use your graphing calculator to verify the derivatives you find throughout this chapter.

■ Example 5

If $y = f(x) = x^4$, then

$$f'(1) = \lim_{x\to 1}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1}\frac{x^4-1}{x-1} = \lim_{x\to 1}\frac{(x-1)(x+1)(x^2+1)}{x-1}$$

$$= \lim_{x\to 1}(x+1)(x^2+1) = 4$$

The notation $f'(1) = 4$ describes the situation precisely: We have evaluated the function f' at the point 1 to obtain the number 4. The other symbols mentioned above are not as useful in this context. For example, if we were to write $y' = 4$ or $D_x y = 4$, the reader might infer that the derivative is the constant function $f'(x) = 4$, whereas the correct formula is $f'(x) = 4x^3$ (Confirm!) Some writers introduce notation like

$$y'\big|_{x=1} = 4 \qquad \text{or} \qquad D_x y\big|_{x=1} = 4$$

to represent the situation, but the functional notation $f'(1) = 4$ is simpler. ■

■ Example 6

The derivative of $y = f(x) = (2x - 1)^2$ is

$$f'(x) = 4(2x-1)$$

(as you can check). The same message is conveyed by

$$\frac{dy}{dx} = 4(2x-1) \qquad \text{or} \qquad D_x y = 4(2x-1)$$

but the notation $y' = 4(2x - 1)$ is risky in some situations (because the symbol y' makes no reference to the independent variable). Suppose, for example, that we substitute $u = 2x - 1$ in $y = (2x - 1)^2$ to obtain $y = u^2$. The derivative of $y = u^2$ *as a function of* u is $dy/du = 2u$ or $D_u y = 2u$ (Example 2, Section 2.1). If we were to write $y' = 2u$, the reader would have a right to complain. For there is a distinction between $y' = 4(2x - 1)$, the derivative of y as a function of x, and $y' = 2u$, the derivative of y as a function of u. It is for this reason, among others, that mathematicians have developed the following terminology, which sometimes strikes the beginner as jargon.

The symbols *dy/dx* and $D_x y$ are translated *derivative of y with respect to x.* To *differentiate y with respect to x* means to find the derivative of $y = f(x)$, and the process of finding it is called *differentiation.*

It is also common practice to regard the symbols d/dx and D_x as *operators* (the operation being differentiation). Whatever is written immediately following them is the function being differentiated. For example,

$$\frac{d}{dx}(x^2) = 2x \quad \text{(Example 2, Section 2.1)}$$

and

$$D_x(x^{3/2}) = \frac{3}{2}\sqrt{x} \quad \text{(Example 3)}$$

One advantage of this notation is that the expression in parentheses need not be identified by some other symbol, say y or $f(x)$. That saves writing. Do not, however, mix the notation! Sometimes students write

$$\frac{dy}{dx}(x^2) = 2x$$

which is confused. We should either let $y = x^2$ and say that $dy/dx = 2x$, or forget about y and simply write

$$\frac{d}{dx}(x^2) = 2x$$

■

Problem Set 2.4

In each of the following, find the indicated derivative by setting up the difference quotient associated with the function at the specified point and then evaluating the appropriate limit.

1. $f'(0)$ if $f(x) = x^2 - 4$
   How could the result have been predicted from the graph?
2. $g'(2)$ if $g(x) = x^2 - 2x$
3. $\phi'(-1)$ if $\phi(t) = 3t^2 - t$
4. $H'(2)$ if $H(x) = 8 - x^3$
5. $f'(0)$ if $f(x) = x^3 + 5$
6. $g'(1)$ if $g(v) = v^6$
7. $p'(0)$ if $p(t) = t\sqrt{t+2}$
8. $F'(1)$ if $F(x) = (x-1)\sqrt{x}$
9. $p'(1)$ if $p(t) = \dfrac{1}{t-2}$
10. $g'(0)$ if $g(x) = \dfrac{1}{x^2+1}$
    In view of the result, how would you draw the graph through (0, 1)?
11. $f'(0)$ if $f(x) = x^{2/3}$
    In view of the result, how would you draw the graph through (0, 0)?
12. $F'(0)$ if $F(x) = \sqrt[3]{x}$
    In view of the result, how would you draw the graph through (0, 0)?
13. $G'(1)$ if $G(x) = 1/\sqrt{x}$
14. $f'(1)$ if $f(t) = \sqrt{1-t}$
    How could the result have been predicted from the graph?
15. The difference quotient associated with $f(x) = \sqrt{25 - x^2}$ at 3 is

    $$Q(x) = \frac{\sqrt{25-x^2} - 4}{x - 3}$$

    and its limit as $x \to 3$ is $f'(3)$. Use a calculator to evaluate $Q(3.1)$, $Q(3.01)$, and $Q(3.001)$, thus obtaining successive approximations to $f'(3)$.
16. Repeat Problem 15 by evaluating $Q(2.9)$, $Q(2.99)$, and $Q(2.999)$.
17. Let $Q(x)$ be the difference quotient associated with $f(x) = (x^2+1)^4$ at 1. Approximate $f'(1)$ by evaluating $Q(1.1)$, $Q(1.01)$, and $Q(1.001)$.

18. Let $Q(x)$ be the difference quotient associated with $f(x) = \sin x$ at 0. Approximate $f'(0)$ by evaluating $Q(\pm 0.1)$, $Q(\pm 0.01)$, and $Q(\pm 0.001)$. *Note*: The angle selector on your calculator must be set on radians. (Why?)

19. Repeat Problem 18 in the case of $f(x) = \cos x$.

20. Repeat Problem 18 in the case of $f(x) = 2^x$. *Note*: You may not know how 2^x is defined for irrational values of x. But a calculator will supply approximations (if it has a y^x key).

21. Repeat Problem 18 in the case of $f(x) = e^x$. *Note*: e is the base of natural logarithms mentioned in Section 1.1. Use the e^x key on your calculator.

In each of the following problems, find the derivative at x_0 as the limit of the difference quotient in the form

$$\frac{f(x) - f(x_0)}{x - x_0}$$

Then drop the subscript to obtain a formula for $f'(x)$.

22. $f(x) = 2x - 3$

23. $f(x) = x^2 - 2x$

24. $f(x) = x^2 + x - 2$

25. $f(x) = 2x^4 + x$

26. $f(x) = \dfrac{5}{x}$

27. $f(x) = \dfrac{2}{x^2}$

28. $f(x) = \dfrac{x - 1}{x}$

29.–35. For the functions in Problems 22 through 28, find $f'(x)$ as the limit of the difference quotient in the form

$$\frac{f(z) - f(x)}{z - x}$$

36.–42. For the functions in Problems 22 through 28, find $f'(x)$ as the limit of the difference quotient in the form

$$\frac{f(x + h) - f(x)}{h}$$

In each of the following problems, find the indicated derivative as the limit of an appropriate difference quotient. Use the form of the difference quotient that seems most convenient.

43. $f'(x)$ if $f(x) = x^2 + x$

44. $g'(x)$ if $g(x) = 5x^2 - x$

45. dy/dx if $y = 2x^3 - 7$

46. $D_x y$ if $y = 4 - x^4$

47. ds/dt if $s = t^3 - t + 1$

48. y' if $y = x^4 + x^2 + 2$

49. y' if $y = x^4 - 3x + 1$

50. $D_t u$ if $u = 1/t^2$

51. $f'(x)$ if $f(x) = \dfrac{x}{x + 5}$

52. $F'(x)$ if $F(x) = \dfrac{x - 3}{x}$

53. $g'(x)$ if $g(x) = \sqrt[3]{x}$

54. $f'(x)$ if $f(x) = x^{2/3}$

55. $p'(t)$ if $p(t) = \sqrt{1 - t}$

56. Use the definition to show that the derivative of a constant function is the zero function. How could this have been predicted from a graph?

57. Use the definition to show that the derivative of a linear function, $f(x) = mx + b$, is $f'(x) = m$. How could this have been predicted from the graph?

58. The graph of $y = |x|$ is shown in Figure 5, Section 2.1. What is its slope when $x > 0$? when $x < 0$? Use your answers to explain the formula

$$D_x|x| = \frac{x}{|x|}$$

What is the domain of this derivative?

59. A painless way to differentiate $f(x) = \sqrt{1 - x^2}$ is to interpret the derivative as slope.

(a) What is the graph of f?

(b) Reasoning from the graph, what would you expect $f'(x)$ to be at $x = 0$? at $x = \pm 1$?

(c) If (x, y) is any interior point of the graph, explain why the slope of the tangent at (x, y) is $-x/y$. Why does it follow that

$$f'(x) = \frac{-x}{\sqrt{1 - x^2}}?$$

What is the domain of f'?

60. Use your graphing calculator to gather "empirical evidence" in support of the result obtained in Problem 59 (c). Enter $y = f(x)$ and use your first derivative option to find the values

$$f'\left(\frac{1}{2}\right) \quad \text{and} \quad f'\left(-\frac{1}{2}\right)$$

Then compare these values to those you obtain by evaluating the derivative function given in Problem 59.

61. Use the *drawing* feature of your graphing calculator to directly draw a tangent to the curve

$$y = \frac{x}{x^2 + 1}$$

at $x = 1$. Visually estimate the slope of this tangent. Use your first derivative option to verify that the slope of this line is 0. Where else on the graph does the tangent line appear to be horizontal?

62. Use the *drawing* feature of your graphing calculator to directly draw a tangent to the curve

$$y = \frac{x}{x^2 + 1}$$

at $x = 2$. Visually estimate the slope of this tangent. Use your first derivative option to verify that the slope of this line is $-\frac{3}{25}$. Is there any other point on the graph with a tangent line parallel to this one?

2.5 Differentiation of Sine and Cosine

In Example 4 of the last section we discussed the problem of finding the derivative of $f(x) = \sin x$. A convenient form of the difference quotient turned out to be

$$q(h) = \frac{f(x+h) - f(x)}{h} = \frac{\sin(x+h) - \sin x}{h} = \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \sin x \left(\frac{\cos h - 1}{h}\right) + \cos x \left(\frac{\sin h}{h}\right)$$

Assuming that the expressions $(\cos h - 1)/h$ and $(\sin h)/h$ have limits as $h \to 0$, we may use the algebra of limits to write

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= (\sin x) \lim_{h\to 0} \left(\frac{\cos h - 1}{h}\right) + (\cos x) \lim_{h\to 0} \left(\frac{\sin h}{h}\right)$$

But what are the values of these limits? Each takes the form 0/0 when 0 is substituted for h; evidently something more clever is needed.

The limit of $(\sin h)/h$ turns out to be the crucial one. Once its value is known, it is an easy matter to find the limit of $(\cos h - 1)/h$, as we will see. Then we will know what the derivative of $\sin x$ is. All the other trigonometric functions can be differentiated in terms of sine (as we will show in the next chapter), so the whole calculus of these functions flows from this limit.

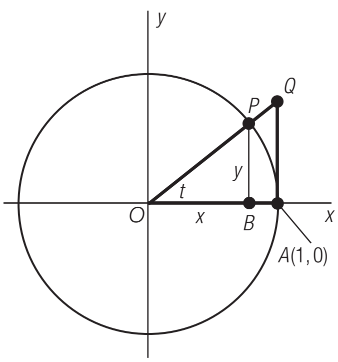


Figure 1 $\cos t \sin t < t < \dfrac{\sin t}{\cos t}$

To investigate the limit, look at the unit circle. Assuming that $0 < t < \pi/2$ (Figure 1), observe that

$$\text{area of } \triangle OBP < \text{area of sector } OAP < \text{area of } \triangle OAQ \tag{1}$$

The area of $\triangle OBP$ is

$$\frac{1}{2}xy = \frac{1}{2}\cos t \sin t \qquad \text{(by definition of sine and cosine)}$$

The area of a circular sector of radius r and central angle θ (in radians) is $A = \frac{1}{2}r^2\theta$. (See Problem 15, Section 1.6.) Hence the area of sector OAP is $\frac{1}{2}t$. The area of $\triangle OAQ$ is

$$\frac{1}{2}(OA)(AQ) = \frac{1}{2}(1)(\tan t) = \frac{1}{2}\frac{\sin t}{\cos t} \qquad \text{(Why?)}$$

Thus the inequalities in (1) become

$$\frac{1}{2}\cos t \sin t < \frac{1}{2}t < \frac{1}{2}\frac{\sin t}{\cos t}$$

Dividing by $\frac{1}{2}\sin t$ (which is positive because $0 < t < \pi/2$), we obtain

$$\cos t < \frac{t}{\sin t} < \frac{1}{\cos t}$$

or (taking reciprocals)

$$\frac{1}{\cos t} > \frac{\sin t}{t} > \cos t \qquad \text{(Order Property 7, Section 1.1)} \qquad (2)$$

Now suppose that in the beginning we had assumed $-\pi/2 < t < 0$ instead of $0 < t < \pi/2$. Then $-t$ would be between 0 and $\pi/2$, and the preceding argument, with t replaced by $-t$, would yield

$$\frac{1}{\cos(-t)} > \frac{\sin(-t)}{-t} > \cos(-t)$$

This reduces to (2), because $\sin(-t) = -\sin t$ and $\cos(-t) = \cos t$. Hence (2) is true for all $t \neq 0$ between $-\pi/2$ and $\pi/2$, whether positive or negative.

Evidently (2) is the heart of the matter. These inequalities show that $(\sin t)/t$ is boxed in between $\cos t$ and $1/\cos t$. When $t \to 0$, both $\cos t$ and $1/\cos t$ approach 1, so by the Squeeze Play Theorem we find that

$$\lim_{t\to 0} \frac{\sin t}{t} = 1$$

Note that in the preceding argument t is simply a real number—the radian measure of angle *AOP*. It is important to realize that if angle *AOP* were measured in degrees, the above limit would no longer be 1 , but $\pi/180$. Since we use the limit to find the derivative of sine (see Example 2), everything in calculus that involves this derivative is predicated on the assumption that the independent variable is either a real number or an angle measured in radians. That is why the analytic trigonometry described in Section 1.6 is indispensable in calculus.

If you have a calculator, you might check the entries in the following table, which indicates how $(\sin t)/t$ approaches 1 as $t \to 0$. Be sure to put your calculator in radian mode! If you set it on degrees, you will find the limit to be $\pi/180 = 0.0174532925199\ldots$.

Convergence of $(\sin t)/t$ to 1 as $t \to 0$

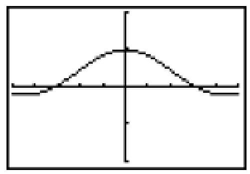
t	$\frac{\sin t}{t}$
± 0.5	0.959
± 0.4	0.974
± 0.3	0.985
± 0.2	0.993
± 0.1	0.998
± 0.01	0.99998
± 0.001	0.9999998
⋮	⋮

GRAPHING CALCULATOR CONCEPTS

Visualizing $\frac{\sin t}{t}$

Use your graphing calculator to look at the graph of

$$y = \frac{\sin x}{x}$$



Identify the viewing window chosen. Why does it look as though the function is defined at $x = 0$ (see GRAPHING CALCULATOR CONCEPTS—Graphing "Holes" on p. 68)? Can you change the window to make the "hole" appear?

■ Example 1

The other limit we need to know to find the derivative of sin x is

$$\lim_{h\to 0}\frac{\cos h - 1}{h}$$

Evaluate this limit.

Solution

Multiplying numerator and denominator by $\cos h + 1$, we have

$$\frac{\cos h - 1}{h} = \frac{(\cos h - 1)(\cos h + 1)}{h(\cos h + 1)} = \frac{\cos^2 h - 1}{h(\cos h + 1)}$$

$$= \frac{-\sin^2 h}{h(\cos h + 1)} \quad (\text{because } \sin^2 h + \cos^2 h = 1)$$

$$= -\left(\frac{\sin h}{h}\right)\left(\frac{\sin h}{\cos h + 1}\right)$$

When $h\to 0$, the first of these fractions approaches 1, as we have just shown. The second fraction approaches 0 because $\sin h\to 0$ and $\cos h\to 1$. Hence

$$\lim_{h\to 0}\frac{\cos h - 1}{h} = 0$$

■

■ Example 2

Show that the derivative of sin x is cos x.

Solution

It is just a matter of putting together what we know. If $f(x) = \sin x$, the difference quotient associated with f at x is

$$q(h) = \sin x\left(\frac{\cos h - 1}{h}\right) + \cos x\left(\frac{\sin h}{h}\right)$$

as we showed at the beginning of this section. Since

$$\lim_{h\to 0}\frac{\cos h - 1}{h} = 0 \quad \text{and} \quad \lim_{h\to 0}\frac{\sin h}{h} = 1$$

we find that

$$f'(x) = \lim_{h\to 0} q(h) = (\sin x)(0) + (\cos x)(1) = \cos x$$

■

In the problem set you are asked to use a similar argument to show that the derivative of cos x is $-\sin x$. The results are fundamental in calculus involving the trigonometric functions:

$$D_x(\sin x) = \cos x \quad \text{and} \quad D_x(\cos x) = -\sin x$$

The fact that

$$\lim_{t\to 0} \frac{\sin t}{t} = 1$$

is important in other parts of calculus as well as in the differentiation of sine and cosine. The following examples illustrate how it may be used to evaluate other limits.

■ Example 3

Evaluate

$$\lim_{t\to 0} \frac{\sin 2t}{t}$$

Solution

Multiply numerator and denominator by 2 to obtain

$$\begin{aligned}
\lim_{t\to 0} \frac{\sin 2t}{t} &= \lim_{t\to 0} \frac{2\sin 2t}{2t} \\
&= \lim_{x\to 0} 2\left(\frac{\sin x}{x}\right) \qquad (x = 2t,\ x\to 0 \text{ when } t\to 0) \\
&= 2\cdot \lim_{x\to 0} \frac{\sin x}{x} = 2\cdot 1 = 2
\end{aligned}$$

■

■ Example 4

Evaluate

$$\lim_{t\to 0} t^2 \csc t$$

Solution

$$\begin{aligned}
\lim_{t\to 0} t^2 \csc t &= \lim_{t\to 0} \frac{t^2}{\sin t} = \lim_{t\to 0} t\left(\frac{t}{\sin t}\right) \\
&= \lim_{t\to 0} t\left(\frac{\sin t}{t}\right)^{-1} \\
&= \lim_{t\to 0} t \cdot \lim_{t\to 0} \left(\frac{\sin t}{t}\right)^{-1} \qquad \text{(algebra of limits)} \\
&= \lim_{t\to 0} t \cdot \left(\lim_{t\to 0} \frac{\sin t}{t}\right)^{-1} \qquad \text{(Composite Function Theorem)} \\
&= 0\cdot 1 = 0
\end{aligned}$$

■

Problem Set 2.5

1. Setting your calculator on degrees, compute $(\sin t)/t$ for $t = 0.1$, 0.01, and 0.001. What number does $(\sin t)/t$ appear to be approaching?

2. Repeat Problem 1 with your calculator set on radians.

3. Setting your calculator on radians, compute $t/\sin 2t$ for $t = 0.1$, 0.01, and 0.001. What number does $t/\sin 2t$ appear to be approaching?

4. Setting your calculator on radians, compute $t \cot t$ for $t = 0.1$, 0.01, and 0.001. What number does $t \cot t$ appear to be approaching?

Use the fact that $\lim_{t\to 0} (\sin t)/t = 1$ to evaluate each of the following limits.

5. $\lim_{t\to 0} \dfrac{t}{\sin t}$

6. $\lim_{t\to 0} \dfrac{\sin 3t}{t}$

7. $\lim_{t\to 0} \dfrac{\sin \pi t}{t}$

8. $\lim_{t\to 0} t \cot t$ Is this equal to $(\lim_{t\to 0} t)(\lim_{t\to 0} \cot t)$?

9. $\lim_{t\to 0} \dfrac{\tan t}{t}$

10. $\lim_{t\to 0} \dfrac{2t - 3\sin t}{t}$

11. $\lim_{t\to 0} \dfrac{2(1 - \cos t)}{t}$

12. $\lim_{t\to 0} \dfrac{\sec t - 1}{t \sec t}$

13. $\lim_{t\to 0} \dfrac{\sin \pi t}{\sin t}$

14. $\lim_{t\to 0} \dfrac{\sin 5t}{\sin 3t}$

15. $\lim_{t\to \pi/4} \dfrac{\cos t - \sin t}{1 - \tan t}$

In each of the following, find the slope of the graph of $y = f(x)$ at the indicated point. Where possible, use the formulas $D_x(\sin x) = \cos x$ and $D_x(\cos x) = -\sin x$; otherwise use the definition of derivative as the limit of a difference quotient.

16. $f(x) = \sin x$ at $x = \pi/2$
How could the result have been predicted from the graph?

17. $f(x) = 3\cos x$ at $x = 0$
How could the result have been predicted from the graph?

18. $f(x) = \cos x$ at $x = \dfrac{3\pi}{2}$

19. $f(x) = \sin x$ at $x = \dfrac{\pi}{3}$

20. $f(x) = \cos x$ at $x = \dfrac{\pi}{6}$

21. $f(x) = \cos 2x$ at $x = 0$

In each of the following, the position of a moving object at time t is given. Find the velocity at the indicated time.

22. $p(t) = \cos t$ at $t = \dfrac{\pi}{2}$

23. $p(t) = \sin t$ at $t = \pi$

24. $p(t) = 2\tan t$ at $t = 0$

25. $p(t) = \tan 2t$ at $t = 0$

26. Show that the difference quotient associated with $f(x) = \cos x$ at x is
$$q(h) = \cos x\left(\frac{\cos h - 1}{h}\right) - \sin x\left(\frac{\sin h}{h}\right)$$
and use the result to prove that $D_x(\cos x) = -\sin x$.

27. Show that the difference quotient associated with $f(x) = \tan x$ at x is
$$q(h) = \frac{\sec^2 x}{1 - \tan x \tan h}\left(\frac{\tan h}{h}\right)$$
and use the result to find $D_x(\tan x)$. *Hint*: Use the addition formula for $\tan(u + v)$ and refer to Problem 9.

28. Show that $D_x(\sin 3x) = 3\cos 3x$.

29. Show that $D_x\left(\cos \dfrac{x}{2}\right) = -\dfrac{1}{2}\sin \dfrac{x}{2}$.

30–33. Verify your answers for Problems 22–25 above using your graphing calculator.

34. Use the *drawing* feature of your graphing calculator to directly draw a tangent to the curve
$$y = \frac{\sin x}{x}$$
at $x = \pi$. Visually estimate the slope of this tangent. Use your first derivative option to verify that the slope of this line is $-\frac{1}{\pi}$.

Additional Problems

In each of the following, find an equation of the tangent to the graph at the given point.

1. $y = 4 - x^2$ at $(2, 0)$
2. $y = \dfrac{x}{x-1}$ at $(2, 2)$
3. $y = \sqrt{x-2}$ at $(2, 0)$
4. $y = \sqrt[3]{x}$ at $(0, 0)$
5. $y = x^{5/3}$ at $(0, 0)$
6. $y = \cos x$ at $(\pi/2, 0)$
7. $y = 3 \sin x$ at $(0, 0)$

In each of the following, s is the position of a moving object at time t. Find the velocity at the given instant.

8. $s = t^3 - t$ at 1
9. $s = 4/t^2$ at 2
10. $s = \sin 2t$ at 0
11. $s = \tan t$ at 0
12. A ball is thrown straight upward. After t seconds its height above the ground (in feet) is $h = 96t - 16t^2$.
    (a) Show that the velocity of the ball at time t is $v(t) = 96 - 32t$.
    (b) When does the ball reach its highest point and how high does it rise?
    (c) When does the ball return to the ground and what is its velocity then?
13. A ball is thrown straight upward from the top of a tower. After t seconds its height above the ground (in feet) is $h = 48 + 32t - 16t^2$.
    (a) Show that the velocity of the ball at time t is $v(t) = 32 - 32t$.
    (b) When does the ball reach its highest point and how high (above the ground) does it rise?
    (c) When does the ball return to the top of the tower? with what speed?
14. If the position of a moving object at time t is $s = \cos t$, how fast is the object moving when $t = 2\pi/3$? in what direction?

Evaluate each of the following limits. Then use an appropriate definition of limit to prove that your answer is correct.

15. $\lim_{x\to 1} (16 - 9x)$
16. $\lim_{x\to 5} (x^2 - 25)$
17. $\lim_{x\to -1} (8 - 3x^2)$
18. $\lim_{x\to 3} (x^3 - 20)$
19. $\lim_{x\to 0} (1 - 2x^3)$
20. $\lim_{x\to 0} \sqrt[3]{x+8}$
21. $\lim_{x\to 1} \left(1 - \dfrac{5}{x}\right)$
22. $\lim_{x\to 0} \left(\cos \dfrac{x}{2} - 1\right)$

Use properties of limits to evaluate each of the following.

23. $\lim_{x\to 1} (x^2 - 6x + 10)$
24. $\lim_{x\to 2} \dfrac{x^2+1}{x^2-1}$
25. $\lim_{x\to 0} x \sin \dfrac{10}{x}$
26. $\lim_{x\to 0} x^2 \cos \dfrac{3}{x^2}$
27. $\lim_{x\to 25} \dfrac{\sqrt{x}-5}{x-25}$
28. $\lim_{x\to 6^+} \dfrac{6-x}{|x-6|}$
29. $\lim_{x\to 1^-} f(x)$, where $f(x) = \begin{cases} 1 - x^2 \text{ if } |x| \le 1 \\ 1 + x \text{ if } |x| > 1 \end{cases}$
30. What are the coordinates of the hole in the graph of
$$y = \frac{x^3-8}{x-2}?$$
31. Evaluate
$$\lim_{t\to 0} \frac{t}{\cot t}$$
    Is it legitimate to write
$$\lim \frac{t}{\cot t} = \frac{\lim t}{\lim \cot t}$$
    in this case? Explain.
32. Evaluate
$$\lim_{x\to 0^+} \frac{|x|}{x} \sin x \quad \text{and} \quad \lim_{x\to 0^-} \frac{|x|}{x} \sin x$$
    Does the limit as $x \to 0$ exist?
33. Evaluate
$$\lim_{x\to 0} \frac{\sin x + 2x}{x}$$

In each of the following, find $f'(x)$ by evaluating the limit of an appropriate difference quotient.

34. $f(x) = x^2 + x$
35. $f(x) = x^2 - 3x$
36. $f(x) = x^4 - 3x$
37. $f(x) = \dfrac{x}{x-5}$
38. $f(x) = \dfrac{x-2}{x}$
39. $f(x) = x^{4/3}$
40. $f(x) = x^{3/4}$
41. Use the definition of derivative to show that if $f(x) = 3x - x^3$, then $f'(x) = 3(1 - x^2)$. Find where the graph of f is rising and where it is falling and sketch it.
42. Use the definition of derivative to show that if $f(x) = x^3 - 12x$, then $f'(x) = 3(x^2 - 4)$. Find where the graph of f is rising and falling and sketch it.
43. Use the interpretation of derivative as slope to explain why the derivative of $y = \sqrt{a^2 - x^2}$ is
$$\frac{dy}{dx} = \frac{-x}{\sqrt{a^2 - x^2}}$$
    Where is the function $f(x) = \sqrt{a^2 - x^2}$ differentiable?

44. If $f(x) = \cos x$, evaluate

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x}$$

by making use of the definition of derivative.

45. If $f(x) = \sin x$, evaluate

$$\lim_{x \to \pi} \frac{f(x) - f(\pi)}{x - \pi}$$

by making use of the definition of derivative.

In each of the following, find $f'(t)$.

46. $f(t) = \sin t \csc t$

47. $f(t) = \cos t \sec t$

48. $f(t) = \sin(-t)$

49. $f(t) = \cos(-t)$

50. $f(t) = \sin(t + \pi)$

51. $f(t) = \cos(\pi - t)$

52. If $f(t) = \sin 2t$, show that $f'(t) = 2 \cos 2t$.

53. If $f(t) = \cos 2t$, show that $f'(t) = -2 \sin 2t$.