CHAPTER 10

Techniques of Integration

Common integration is only the memory of differentiation.

AUGUSTUS DE MORGAN (1806–1871)

Mathematicians are like Frenchmen: whatever you say to them they translate into their own language, and forthwith it is something entirely different.

JOHANN WOLFGANG VON GOETHE (1749–1832)

One should not use an elephant gun to shoot a mouse.

ANONYMOUS

COMPUTER CALCULUS: INTEGRATING WITH TECHNOLOGY

In this chapter you will learn a variety of techniques for evaluating integrals. Many of these techniques are quite ingenious, demonstrating the power of calculus problems to stimulate critical thinking. It would be a sad day when mathematics professors stopped teaching students how to do these problems with pencil and paper. (For an imaginative picture of a future when people have forgotten that it is even possible to "do math in your head," read Isaac Asimov's famous 1957 short story "The Feeling of Power," copyrighted by the Quinn Publishing Co., Inc.)

However, it is not sad at all to have access to powerful software like Maple or Mathematica. These CAS (computer algebra systems) can yield results for integrals which would be tedious, and not necessarily instructive, to do by hand. It is also important to remember that many integrals are far from elementary. It is good to have computers!

Problem 11 in Problem Set 10.2 asks you to find the integral $\int \sin^3 x \, dx$. You can accomplish this by cleverly rewriting the sine cubed as a product, using a Pythagorean identity to introduce cosine, then using a substitution. This will yield the answer given in the Answers section at the back of the book. Maple yields an answer that looks different.

Input the following syntax:

```
int((sin(x)^3,x);
```

The output is

$-1/3 \sin(x)\hat{}2 \cos(x) - 2/3 \cos(x)$, which we might write as

$$-\frac{1}{3}\sin^2 x \cos x - \frac{2}{3}\cos x$$

Can you use trigonometric identities to show that this result is the same as the answer we found? Can you see why it may be unwise to leave mathematics strictly to the computers and fail to gain an understanding and appreciation of it ourselves? Read the story!

■ ■ ■

THIS CHAPTER IS devoted to what might be called the *art of integration*. We say art (rather than science) because it is hard to say anything very systematic about the methods used to tackle a given integral. Differentiation is straightforward; antidifferentiation often requires some imagination.

There are really only two general approaches to integration (although it may not look that way to the beginner). One is to make a substitution that reduces the integral to a familiar standard form; the other is *integration by parts* (the first topic of this chapter). From your experience with substitution in the last four chapters, you are undoubtedly aware of the central problem. *It is essential to perceive the standard form lurking behind an integral in order to know what substitution to try.* That is why we have been collecting integration formulas as though they were rare jewels.

You will find a list of previously derived results on the inside back cover of the book; they will be needed from here on out. Whether they should be memorized (in whole or in part) is a question we prefer to leave open. One can certainly argue that they need not be, on the grounds that we can always look them up. On the other hand, how do we recognize a good substitution if we are ignorant of the standard forms? It is no accident that most students who master the art of integration seem to know these formulas pretty well (whether required to learn them or not).

You will also find a list of hyperbolic integrals on the inside back cover (not given before) that are analogous to familiar trigonometric formulas. While they are not so often used, they are worth listing for reference (and are easily checked). Two of them (the integrals of sech and csch) are surprising; you might enjoy figuring out how they are obtained.

Although this list is a good start on a table of integrals, you should realize that no table is adequate for all integration problems. A simple example is

$$\int_0^1 e^{-x^2}\,dx = ?$$

No elementary function exists whose derivative is e^{-x^2}; in such a case it is necessary to resort to approximation techniques. We will take up that question at the end of the chapter.

10.1 Integration by Parts

The Product Rule for derivatives says that if u and v are functions of x, then

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + \frac{du}{dx}v$$

or (in differential form)

$$d(uv) = u\,dv + v\,du$$

Integrating both sides of this equation, we have $uv = \int u\,dv + \int v\,du$, from which

$$\int u\,dv = uv - \int v\,du$$

This innocent-looking formula is the source of a powerful method of integration known as *integration by parts*. A few examples will indicate how useful it is.

■ Example 1

Find $\int xe^x\,dx$.

Solution

We think of the integrand $xe^x\,dx$ as having two "parts," naming one part u and the other dv. This may be done in many ways, one of which is $u = x$ and $dv = e^x\,dx$. Calculating

$$du = dx \qquad \text{and} \qquad v = \int e^x\,dx = e^x \qquad \text{(arbitrary constant omitted)}$$

we find

$$\int xe^x\,dx = \int u\,dv = uv - \int v\,du = xe^x - \int e^x\,dx = xe^x - e^x + C$$

Another choice of parts in this problem is $u = e^x$ and $dv = x\,dx$, from which $du = e^x\,dx$ and $v = x^2/2$. The parts formula gives

$$\int xe^x\,dx = \frac{1}{2}x^2e^x - \frac{1}{2}\int x^2e^x\,dx$$

While this is certainly correct, it is not going anywhere. The new integral (on the right side) is harder than the original one. This illustrates the fact that integration by parts is not a routine procedure; it requires some judgment (based on experience) of what choices of u and dv are good ones. ■

Remark

It is worth noting that once u is chosen in the method of integration by parts, the choice of dv is automatic. Thus in Example 1 we chose $u = x$; then we had no choice but to write $dv = e^x\,dx$. The calculation of $du = dx$ (by differentiation of $u = x$) and of $v = e^x$ (by integration of $dv = e^x\,dx$) is also automatic. In view of these facts the choice of u is clearly crucial. There are two critical considerations to keep in mind when this choice is made:

1. We must be able to evaluate $\int dv$ (to obtain v from dv).
2. The new integral $\int v\,du$ must be easier to evaluate than the original integral $\int u\,dv$. (See Example 7 for an exception to this statement, however.)

■ Example 2

You may have already encountered the formula

$$\int \ln x\,dx = x\ln x - x + C \qquad \text{(Additional Problem 17, Chapter 8)}$$

To see where it comes from, let $u = \ln x$ and $dv = dx$. Then $du = dx/x$ and $v = x$, from which

$$\int \ln x\,dx = uv - \int v\,du = x\ln x - \int dx = x\ln x - x + C$$ ■

■ Example 3

To find $\int \sin^{-1} x\,dx$ (Problem 35, Section 9.2), let $u = \sin^{-1} x$ and $dv = dx$. Then

$$du = \frac{dx}{\sqrt{1-x^2}} \qquad \text{and} \qquad v = x$$

from which

$$\int \sin^{-1} x\,dx = x\sin^{-1} x - \int \frac{x\,dx}{\sqrt{1-x^2}}$$

The new integral may be found by making a substitution, namely $u = 1 - x^2$, $du = -2x\,dx$. Thus

$$\int \frac{x\,dx}{\sqrt{1-x^2}} = -\frac{1}{2}\int u^{-1/2}\,du = -\frac{1}{2}\cdot 2u^{1/2} = -\sqrt{1-x^2}$$

and we find

$$\int \sin^{-1} x\,dx = x\sin^{-1} x + \sqrt{1-x^2} + C$$ ■

Remark

The letter u occurs in Example 3 with different meanings. Such repetition is convenient (if the earlier u is no longer part of the problem), but of course it should not be done if there is any danger of confusion.

■ Example 4

To find

$$\int_0^{\pi/2} x^2 \sin x\,dx$$

let $u = x^2$ and $dv = \sin x\,dx$. Then $du = 2x\,dx$ and $v = -\cos x$, from which

$$\int x^2 \sin x\,dx = -x^2\cos x + 2\int x\cos x\,dx$$

While the problem is not solved, the new integral is simpler than the original one; we find it by using integration by parts again:

$$u = x,\ dv = \cos x\,dx \qquad (\text{hence } du = dx,\ v = \sin x)$$

Then

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x$$

The original integral is

$$\int_0^{\pi/2} x^2 \sin x\, dx = (-x^2 \cos x + 2x \sin x + 2 \cos x)\Big|_0^{\pi/2} = \pi - 2$$ ■

Repeated integration by parts (as in Example 4) occurs so often that it is worthwhile to develop a formula for it. Suppose that we want to find

$$\int f(x)g(x)\, dx$$

where f is a polynomial of degree n and g is a function that can be integrated repeatedly. Letting

$$u = f(x) \qquad \text{and} \qquad dv = g(x)\, dx$$

we have $du = f'(x)\, dx$ and $v = G_1(x)$ (where G_1 is an antiderivative of g). The parts formula gives

$$\int f(x)g(x)\, dx = f(x)G_1(x) - \int f'(x)G_1(x)\, dx$$

Since f is a polynomial, so is f', and its degree is one less than the degree of f. Thus the new integral may be expected to be simpler than the original one (assuming that G_1 is no harder to integrate than g). Use the parts formula again, this time with

$$u = f'(x) \qquad \text{and} \qquad dv = G_1(x)\, dx$$

Then $du = f''(x)\, dx$ and $v = G_2(x)$ (where G_2 is an antiderivative of G_1). This gives

$$\int f(x)g(x)\, dx = f(x)G_1(x) - \left[f'(x)G_2(x) - \int f''(x)G_2(x)\, dx \right]$$

$$= f(x)G_1(x) - f'(x)G_2(x) + \int f''(x)G_2(x)\, dx$$

We will carry out one more step to clarify the general pattern. Let

$$u = f''(x) \qquad \text{and} \qquad dv = G_2(x)\, dx$$

Then $du = f'''(x)\, dx$ and $v = G_3(x)$ (where G_3 is an antiderivative of G_2), and we have

$$\int f(x)g(x)\, dx = f(x)G_1(x) - f'(x)G_2(x) + f''(x)G_3(x) - \int f'''(x)G_3(x)\, dx$$

This process stops in n steps (when the polynomial has been differentiated down to a constant). Hence the formula we are seeking reads as follows.

Repeated Integration by Parts

$$\int fg = fG_1 - f'G_2 + f''G_3 - f'''G_4 + \dots + (-1)^n f^{(n)} G_{n+1} + C$$

where f is a polynomial of degree n and $G_1, G_2, \dots, G_{n+1}$ are successive antiderivatives of g.

■ Example 5

Use the formula for repeated integration by parts to confirm the result in Example 4.

Solution

With $f(x) = x^2$ and $g(x) = \sin x$, our formula reads

$$\begin{aligned}\int x^2 \sin x\,dx &= (x^2)(-\cos x) - (2x)(-\sin x) + (2)(\cos x) + C\\ &= -x^2\cos x + 2x\sin x + 2\cos x + C\end{aligned}$$

As you can see, this is the same antiderivative we found in Example 4. ■

■ Example 6

Find $\int x^4 e^{2x}\,dx$.

Solution

This would take a while using the ordinary parts formula repeatedly. The repeated integration by parts formula, however, reduces it to an easy problem:

$$\begin{aligned}\int x^4 e^{2x}\,dx &= (x^4)(\tfrac{1}{2}e^{2x}) - (4x^3)(\tfrac{1}{4}e^{2x}) + (12x^2)(\tfrac{1}{8}e^{2x}) - (24x)(\tfrac{1}{16}e^{2x}) + (24)(\tfrac{1}{32}e^{2x}) + C\\ &= \tfrac{1}{4}e^{2x}(2x^4 - 4x^3 + 6x^2 - 6x + 3) + C\end{aligned}$$

■

■ Example 7

To find $\int e^x \cos x\,dx$, let

$$u = e^x \quad \text{and} \quad dv = \cos x\,dx$$

Then $du = e^x\,dx$ and $v = \sin x$, from which

$$\int e^x \cos x\,dx = e^x \sin x - \int e^x \sin x\,dx$$

In the new integral let

$$u = e^x \text{ and } dv = \sin x\,dx \qquad (\text{hence } du = e^x\,dx \text{ and } v = -\cos x)$$

This gives

$$\begin{aligned}\int e^x\cos x\,dx &= e^x \sin x - \left(-e^x\cos x + \int e^x \cos x\,dx\right)\\ &= e^x\sin x + e^x \cos x - \int e^x\cos x\,dx\end{aligned}$$

While it may appear that we are going in circles, in fact we are not. Simply solve for the original integral to obtain

$$2\int e^x\cos x\,dx = e^x\sin x + e^x\cos x \qquad (\text{arbitrary constant omitted})$$

$$\int e^x\cos x\,dx = \tfrac{1}{2}e^x(\sin x + \cos x) + C$$

■

■ Example 8

Derive a "reduction formula" for $\int \sin^n x\, dx$ (where n is a positive integer greater than 1).

Solution

Let $u = \sin^{n-1} x$ and $dv = \sin x\, dx$. Then

$$du = (n-1)\sin^{n-2} x \cos x\, dx \qquad \text{and} \qquad v = -\cos x$$

from which

$$\int \sin^n x\, dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x \cos^2 x\, dx$$

Replacing $\cos^2 x$ by $1 - \sin^2 x$ in the new integral, we have

$$\int \sin^n x\, dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx - (n-1)\int \sin^n x\, dx$$

from which

$$n\int \sin^n x\, dx = -\sin^{n-1} x \cos x + (n-1)\int \sin^{n-2} x\, dx$$

Hence

$$\int \sin^n x\, dx = \frac{-\sin^{n-1} x \cos x}{n} + \frac{n-1}{n}\int \sin^{n-2} x\, dx \quad (n > 1)$$ ■

A special case of this result is worth recording for future use, along with a similar formula we ask you to derive in the problem set:

> If n is a positive integer greater than 1, then
>
> $$\int_0^{\pi/2} \sin^n x\, dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\, dx$$
>
> $$\int_0^{\pi/2} \cos^n x\, dx = \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2} x\, dx$$

For example, when $n = 6$ we have

$$\int_0^{\pi/2} \sin^6 x\, dx = \frac{5}{6}\int_0^{\pi/2} \sin^4 x\, dx$$

$$= \frac{5}{6}\cdot\frac{3}{4}\int_0^{\pi/2} \sin^2 x\, dx \qquad \text{(using the formula with } n = 4\text{)}$$

$$= \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\int_0^{\pi/2} dx \qquad \text{(using it again with } n = 2\text{)}$$

$$= \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{5\pi}{32}$$

Problem Set 10.1

Find a formula for each of the following antiderivatives.

1. $\int xe^{2x}\,dx$

2. $\int (3-x)e^{3x}\,dx$

3. $\int x\sin x\,dx$

4. $\int x\cos 2x\,dx$

5. $\int x\sqrt{x-1}\,dx$

6. $\int x\sqrt{2-x}\,dx$

7. $\int x\ln x\,dx$

8. $\int \ln(x^2+1)\,dx$

9. $\int x^2e^x\,dx$

10. $\int x^2e^{2x}\,dx$

11. $\int \sin(\ln x)\,dx$

12. $\int \cos(\ln x^2)\,dx$

13. $\int \sinh^{-1}x\,dx$

14. $\int x\tan^{-1}x\,dx$

15. $\int \sin x\sin 3x\,dx$

16. $\int \cos 2x\cos 3x\,dx$

17. $\int \sec^3 x\,dx$ *Hint:* Let $u = \sec x$ and $dv = \sec^2 x\,dx$. Then replace $\tan^2 x$ by $\sec^2 x - 1$ in the new integral.

18. $\int \csc^3 x\,dx$

Evaluate each of the following.

19. $\int_0^1 x(2x-1)^3\,dx$

20. $\int_{\pi/6}^{\pi/2} x\csc^2 x\,dx$

21. $\int_0^{\pi} e^{-x}\sin x\,dx$

22. $\int_{e^{-\pi}}^{e^{\pi}} \cos(\ln x)\,dx$

23. $\int_{\pi/4}^{\pi/2} \csc^3 x\,dx$

24. $\int_1^2 \sec^{-1}x\,dx$

25. $\int_0^{1/2} \tanh^{-1}x\,dx$

26. $\int_0^{\pi/6} \sin 2x\cos 3x\,dx$

27. Find $\int e^x\sinh x\,dx$ without using integration by parts.

28. Find $\int e^{-x}\cosh x\,dx$ without using integration by parts.

29. Find the area bounded by the curve $y = \ln x$, the x axis, and the line $x = e$.

30. Find the area bounded by the curve $y = \tan^{-1}x$, the x axis, and the line $x = 1$.

31. The region under the curve $y = \sin x$, $0 \le x \le \pi/2$, is rotated about the y axis. Find the volume generated.

32. The region under the curve $y = e^{-2x}$, $0 \le x \le 1$, is rotated about the y axis. Find the volume generated.

33. Find the centroid of the region under the curve $y = \cos x$, $0 \le x \le \pi/2$.

34. Find the centroid of the region under the curve $y = e^x$, $0 \le x \le 1$.

Use the formula for repeated integration by parts to find each of the following.

35. $\int x^2e^{-x}\,dx$

36. $\int x^3e^{2x}\,dx$

37. $\int (x^2+1)\sin x\,dx$

38. $\int (1-x^2)\cos x\,dx$

39. $\int x^3(x-1)^4\,dx$

40. $\int (x-2)^2(x+2)^3\,dx$

41. Show that if $b \ne 0$,

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} + C$$

What if $b = 0$ (and $a \ne 0$)?

42. Show that if $b \ne 0$,

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(b\sin bx + a\cos bx)}{a^2+b^2} + C$$

What if $b = 0$ (and $a \ne 0$)?

43. Derive the reduction formula

$$\int \cos^n x\,dx = \frac{\cos^{n-1}x\sin x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx$$

where n is a positive integer greater than 1.

44. Use Problem 43 to explain why

$$\int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \cos^{n-2}x\,dx$$

Evaluate each of the following integrals.

45. $\int_0^{\pi/2} \cos^6 x\,dx$

46. $\int_0^{\pi/2} \sin^8 x\,dx$

47. $\int_0^{\pi} \sin^5 \frac{x}{2}\,dx$

48. $\int_0^{\pi} \cos^7 2x\,dx$

49. Derive the famous *Wallis formulas*

$$\int_0^{\pi/2} \sin^{2k}x\,dx = \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots(2k)}\cdot\frac{\pi}{2}$$

$$\int_0^{\pi/2} \sin^{2k+1}x\,dx = \frac{2\cdot 4\cdot 6\cdots(2k)}{1\cdot 3\cdot 5\cdots(2k+1)}$$

where k is any positive integer.

50. It is not hard to prove that the ratio of the integrals in Problem 49 approaches 1 as k increases.

(a) Explain why it follows that

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdots$$

in the sense that $\pi/2$ may be approximated as closely as desired by using sufficiently many factors of this "infinite product."

(b) Show that part (a) can be written in the form

$$\pi = 2 \cdot \frac{4}{3} \cdot \frac{16}{15} \cdot \frac{36}{35} \cdot \frac{64}{63} \cdots$$

(c) Use the formula in part (b) (and a calculator) to get an idea of how fast the product converges to π. (Punch 2 times 4 divided by 3 times 16 divided by 15 times 36 . . . .)

10.2 Integrals Involving Trigonometric Functions

In the next section we will discuss an important class of substitutions that have the effect of transforming certain algebraic integrals into integrals involving trigonometric functions. If we can handle the latter, we are in a position to dispose of many previously intractable problems. Our purpose in this section is to discuss the trigonometric integrals that are most useful in this connection. They are of three types, namely

$$1. \int \sin^m x \cos^n x \, dx \qquad 2. \int \tan^m x \sec^n x \, dx \qquad 3. \int \cot^m x \csc^n x \, dx$$

Rather than considering a systematic listing of cases under these three types, we simply present examples to show you the techniques that are needed for dealing with trigonometric integrals.

■ Example 1

Find $\int \sin^3 x \sqrt{\cos x} \, dx$.

Solution

The odd exponent ($m = 3$) allows us to break off $\sin x \, dx$ and change the remaining even power ($\sin^2 x$) to an expression involving cosine:

$$\int \sin^3 x \sqrt{\cos x} \, dx = \int \sin^2 x \sqrt{\cos x} \cdot \sin x \, dx$$

$$= \int (1 - \cos^2 x) \sqrt{\cos x} \cdot \sin x \, dx$$

Now let $u = \cos x$ and $du = -\sin x \, dx$ to obtain

$$\int \sin^3 x \sqrt{\cos x} \, dx = -\int (1 - u^2) \sqrt{u} \, du = \int (u^{5/2} - u^{1/2}) \, du$$

$$= \frac{2}{7} u^{7/2} - \frac{2}{3} u^{3/2} + C = \frac{2}{7} (\cos x)^{7/2} - \frac{2}{3} (\cos x)^{3/2} + C \quad ■$$

■ Example 2

Find

$$\int_0^{\pi} \sin^2\frac{t}{2}\cos^5\frac{t}{2}\,dt$$

Solution

This time the odd exponent ($n = 5$) suggests breaking off $\cos\frac{t}{2}dt$ and expressing what remains in terms of $\sin\frac{t}{2}$:

$$\begin{aligned}
\int_0^{\pi} \sin^2\frac{t}{2}\cos^5\frac{t}{2}\,dt &= \int_0^{\pi} \sin^2\frac{t}{2}\cos^4\frac{t}{2}\cdot\cos\frac{t}{2}\,dt \\
&= \int_0^{\pi} \sin^2\frac{t}{2}\left(1-\sin^2\frac{t}{2}\right)^2\cdot\cos\frac{t}{2}\,dt \\
&= 2\int_0^1 u^2(1-u^2)^2\,du \qquad \left(u=\sin\frac{t}{2},\ du=\frac{1}{2}\cos\frac{t}{2}\,dt\right) \\
&= 2\int_0^1 (u^6-2u^4+u^2)\,du \\
&= 2\left(\frac{u^7}{7}-\frac{2u^5}{5}+\frac{u^3}{3}\right)\Bigg|_0^1 = \frac{16}{105}
\end{aligned}$$

■

■ Example 3

Find $\int \sin^2 t\cos^2 t\,dt$.

Solution

The even exponents suggest the multiplication formulas

$$\sin^2 t = \tfrac{1}{2}(1-\cos 2t) \qquad \text{and} \qquad \cos^2 t = \tfrac{1}{2}(1+\cos 2t)$$

(See Section 1.6.) We use them to write

$$\int \sin^2 t\cos^2 t\,dt = \frac{1}{4}\int (1-\cos^2 2t)\,dt = \frac{1}{4}\int dt - \frac{1}{4}\int \cos^2 2t\,dt$$

In the last integral we use a multiplication formula again, in the form

$$\cos^2 2t = \frac{1}{2}(1+\cos 4t)$$

This yields

$$\begin{aligned}
\int \sin^2 t\cos^2 t\,dt &= \frac{1}{4}\int dt - \frac{1}{8}\int (1+\cos 4t)\,dt = \frac{1}{8}\int dt - \frac{1}{8}\int \cos 4t\,dt \\
&= \frac{1}{8}t - \frac{1}{32}\sin 4t + C
\end{aligned}$$

■

■ Example 4

Find

$$\int \frac{\sec^4 t}{\tan t}\,dt$$

Solution

The fact that $\sec^2 t$ is the derivative of $\tan t$ suggests that we break off $\sec^2 t\,dt$:

$$\int \frac{\sec^4 t}{\tan t}\,dt = \int \frac{\sec^2 t}{\tan t}\cdot \sec^2 t\,dt$$

The identity $\sec^2 t - \tan^2 t = 1$ enables us to write

$$\int \frac{\sec^4 t}{\tan t}\,dt = \int \frac{\tan^2 t + 1}{\tan t}\cdot \sec^2 t\,dt$$

Now let $u = \tan t$, $du = \sec^2 t\,dt$ to obtain

$$\int \frac{\sec^4 t}{\tan t}\,dt = \int \frac{u^2+1}{u}\,du = \int \left(u + \frac{1}{u}\right)du = \frac{u^2}{2} + \ln|u| + C$$

$$= \frac{1}{2}\tan^2 t + \ln|\tan t| + C$$

■

■ Example 5

Find

$$\int_0^{\pi/3} \tan^3 t \sec t\,dt$$

Solution

This time we break off $\sec t \tan t\,dt$, having in mind the derivative of $\sec t$ and intending to express what remains in terms of $\sec t$:

$$\int_0^{\pi/3} \tan^3 t \sec t\,dt = \int_0^{\pi/3} \tan^2 t \cdot \sec t \tan t\,dt$$

$$= \int_0^{\pi/3} (\sec^2 t - 1)\cdot \sec t \tan t\,dt$$

$$= \int_1^2 (u^2 - 1)\,du \qquad (u = \sec t,\ du = \sec t \tan t\,dt)$$

$$= \frac{4}{3}$$

■

■ Example 6

Find $\int \cot^4 t\,dt$.

Solution

Use the identity $\csc^2 t - \cot^2 t = 1$ to write

$$\int \cot^4 t\, dt = \int \cot^2 t\,(\csc^2 t - 1)\, dt = \int \cot^2 t \cdot \csc^2 t\, dt - \int \cot^2 t\, dt$$

$$= \int \cot^2 t \cdot \csc^2 t\, dt - \int (\csc^2 t - 1)\, dt$$

$$= -\frac{1}{3}\cot^3 t + \cot t + t + C$$

These examples and the problem set should give you the idea. We will put the results to work in the next section.

Problem Set 10.2

Find each of the following integrals.

1. $\int_0^{\pi/2} \sin^4 x \cos^3 x\, dx$
2. $\int_0^{\pi/2} \cos x \sqrt{\sin x}\, dx$
3. $\int \sin^3 x \cos^3 x\, dx$
4. $\int \frac{\cos^4 t}{\sin^2 t}\, dt$
5. $\int_0^{\pi/2} \sin^4 x \cos^2 x\, dx$
6. $\int_0^{\pi} \cos^4 t\, dt$
7. $\int \sin^5 t\, dt$
8. $\int_0^{\pi/2} \cos^3 x\, dx$
9. $\int \sin^2 2x\, dx$
10. $\int \frac{\sin^5 x}{\cos^5 x}\, dx$
11. $\int \sin^3 x\, dx$
12. $\int_0^{\pi/4} \frac{\sin^4 x}{\cos^2 x}\, dx$
13. $\int \sin t \cot t\, dt$
14. $\int \cos^2 t \tan t\, dt$
15. $\int_{\pi/4}^{\pi/3} \frac{\sec^2 x}{1 + \tan x}\, dx$
16. $\int \tan^2 x \sec x\, dx$

    Hint: You will need the formula for $\int \sec^3 x\, dx$ (Problem 17, Section 10.1 and Example 3, Section 10.3).
17. $\int \sec^4 x\, dx$
18. $\int \frac{\tan x}{\sec^2 x}\, dx$
19. $\int (\sec^4 x - \tan^4 x)\, dx$
20. $\int \tan^4 x\, dx$
21. $\int \tan x \sec^3 x\, dx$
22. $\int \tan^2 x \sec^4 x\, dx$
23. $\int_{\pi/4}^{\pi/3} \tan^4 t \sec^4 t\, dt$
24. $\int \tan^3 x\, dx$
25. $\int (\tan x + \cot x)^2\, dx$
26. $\int_0^{\pi/4} \sec^6 t\, dt$
27. $\int \tan^6 t\, dt$
28. $\int \tan^3 x (\sec x)^{3/2}\, dx$
29. $\int_{\pi/4}^{\pi/2} \cot^2 t\, dt$
30. $\int \cot^3 t\, dt$
31. $\int_{\pi/4}^{\pi/2} \cot x \csc^3 x\, dx$
32. $\int_{\pi/4}^{\pi/2} \csc^4 x\, dx$
33. $\int \sec^4 x \csc^4 x\, dx$
34. $\int_{\pi/6}^{\pi/2} \frac{\cot t}{\csc^3 t}\, dt$
35. Work Example 3 by using the formula $\sin 2t = 2 \sin t \cos t$.
36. The formula

    $$\int \sec x \tan x\, dx = \sec x + C$$

    is a standard form. Acting as though you were ignorant of it, derive it by writing the integrand in terms of $\sin x$ and $\cos x$.
37. Derive the reduction formula

    $$\int \tan^n x\, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\, dx$$
38. Find the length of the curve $y = \ln \cos x$, $-\pi/4 \le x \le \pi/4$.
39. Find the volume of the solid generated when the curve $y = \cot^2 x$, $\pi/4 \le x \le 3\pi/4$, is rotated about the x axis.
40. How much surface area is generated when the cycloid $x = 1 - \cos t$, $y = t - \sin t$, $0 \le t \le 2\pi$, is rotated about the y axis?

10.3 Trigonometric Substitutions

Many important integration problems involve expressions of the form

$$\sqrt{a^2 - x^2} \qquad \sqrt{a^2 + x^2} \qquad \sqrt{x^2 - a^2}$$

where a is a (positive) constant. In such cases a *trigonometric substitution* is often helpful, of the type

$$x = a \sin t \qquad x = a \tan t \qquad x = a \sec t$$

respectively. To see what effect these substitutions have on the corresponding expressions, observe that

$$\begin{aligned} a^2 - x^2 &= a^2 - a^2 \sin^2 t = a^2(1 - \sin^2 t) = a^2 \cos^2 t \\ a^2 + x^2 &= a^2 + a^2 \tan^2 t = a^2(1 + \tan^2 t) = a^2 \sec^2 t \\ x^2 - a^2 &= a^2 \sec^2 t - a^2 = a^2(\sec^2 t - 1) = a^2 \tan^2 t \end{aligned}$$

respectively. If we assume that our substitutions are invertible (that is, only restricted trigonometric functions having inverses are used), then

$$\begin{aligned} \sqrt{a^2 - x^2} &= a\,|\cos t| = a \cos t &&\text{(because } -\pi/2 \le t \le \pi/2) \\ \sqrt{a^2 + x^2} &= a\,|\sec t| = a \sec t &&\text{(because } -\pi/2 < t < \pi/2) \\ \sqrt{x^2 - a^2} &= a\,|\tan t| = a \tan t &&\text{(because } 0 \le t < \pi/2 \text{ or } \pi \le t < 3\pi/2) \end{aligned}$$

In other words, each substitution eliminates the radical by replacing it with a trigonometric function. This has a remarkable effect on many otherwise difficult integrals, as you will see.

Remark

A review of Section 9.1 may be needed here. What we are saying is that the substitutions

$$x = a \sin t \qquad x = a \tan t \qquad x = a \sec t$$

are equivalent to

$$t = \sin^{-1} \frac{x}{a} \qquad t = \tan^{-1} \frac{x}{a} \qquad t = \sec^{-1} \frac{x}{a}$$

respectively, provided that sine, tangent, and secant are restricted as described in Section 9.1. You may also want to look at Section 6.5 again, particularly the two substitution theorems stated there (and Example 7). These theorems call for a “change of variable” function $t = h(x)$ that gives the new variable in terms of the old. If the function is invertible, however, it makes no difference, since either x or t can be expressed in terms of the other.

■ Example 1

An important standard form (first stated in Problem 38, Section 9.2, but as yet unexplained) is

$$\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$$

To see where it comes from, make the substitution

$$x = a \sin t \qquad (-\pi/2 \le t \le \pi/2)$$

Then $dx = a \cos t\, dt$ and (as explained above) $\sqrt{a^2 - x^2} = a \cos t$. Our integral becomes

$$\int \sqrt{a^2 - x^2}\, dx = \int a \cos t \cdot a \cos t\, dt = a^2 \int \cos^2 t\, dt$$

$$= \frac{a^2}{2} \int (1 + \cos 2t)\, dt$$

$$= \frac{a^2}{2}\left(t + \frac{1}{2} \sin 2t\right) + C$$

$$= \frac{a^2}{2}(t + \sin t \cos t) + C \qquad (\text{because } \sin 2t = 2 \sin t \cos t)$$

Since our change of variable equation ($x = a \sin t$) is equivalent to

$$t = \sin^{-1} \frac{x}{a}$$

and since

$$\sin t = \frac{x}{a} \qquad \text{and} \qquad \cos t = \frac{\sqrt{a^2 - x^2}}{a}$$

we find

$$\int \sqrt{a^2 - x^2}\, dx = \frac{a^2}{2}\left(\sin^{-1} \frac{x}{a} + \frac{x}{a} \cdot \frac{\sqrt{a^2 - x^2}}{a}\right) + C$$

$$= \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C$$

as advertised. ■

■ Example 2

To find

$$\int_0^3 \frac{x^3\, dx}{\sqrt{9 + x^2}}$$

make the substitution

$$x = 3 \tan t \qquad (-\pi/2 < t < \pi/2)$$

Then $dx = 3 \sec^2 t\, dt$ and

$$\sqrt{9 + x^2} = \sqrt{9 + 9 \tan^2 t} = 3\sqrt{1 + \tan^2 t} = 3\sqrt{\sec^2 t}$$
$$= 3 \sec t \qquad (-\pi/2 < t < \pi/2)$$

Since $x = 0 \Rightarrow t = 0$ and $x = 3 \Rightarrow t = \pi/4$ (why?), we find

$$\int_0^3 \frac{x^3\, dx}{\sqrt{9 + x^2}} = \int_0^{\pi/4} \frac{27 \tan^3 t \cdot 3 \sec^2 t\, dt}{3 \sec t} = 27 \int_0^{\pi/4} \sec t \tan^3 t\, dt$$

At this point we are faced with a new problem (not necessarily easier than the original one, but let's hope for the best). The appearance of secant and tangent together in the integrand is a good sign because we know an identity ($\sec^2 t - \tan^2 t = 1$) and several differentiation and integration formulas involving these functions.

You may have to stare at the problem for a while to see what to do. Let's break off a tangent (to go with secant) and see what happens:

$$\int \sec t \tan^3 t\, dt = \int \tan^2 t \cdot \sec t \tan t\, dt$$

The idea behind this move is that $\sec t \tan t$ is the derivative of $\sec t$; if we can express the rest of the integrand in terms of $\sec t$, the substitution

$$u = \sec t \qquad du = \sec t \tan t\, dt$$

will do the trick. To this end, write $\tan^2 t = \sec^2 t - 1$. Then

$$\int \sec t \tan^3 t\, dt = \int (\sec^2 t - 1) \sec t \tan t\, dt = \int (u^2 - 1)\, du$$

Now it is easy to finish:

$$\int_0^3 \frac{x^3\, dx}{\sqrt{9 + x^2}} = 27 \int_0^{\pi/4} \sec t \tan^3 t\, dt$$

$$= 27 \int_1^{\sqrt{2}} (u^2 - 1)\, du \qquad \left(t = 0 \Rightarrow u = 1 \text{ and } t = \frac{\pi}{4} \Rightarrow u = \sqrt{2}\right)$$

$$= 27\left(\frac{u^3}{3} - u\right)\Bigg|_1^{\sqrt{2}} = 9u(u^2 - 3)\Bigg|_1^{\sqrt{2}}$$

$$= 9[\sqrt{2}\,(2 - 3) - (1 - 3)] = 9(2 - \sqrt{2}\,)$$ ■

Remark

There is an easier way to do Example 2, based on the substitution $u = \sqrt{9 + x^2}$. (See the problem set.) Trigonometric substitution is an important technique of integration, but it is not guaranteed to give the quickest results in all cases that appear to call for it.

■ Example 3

Find a formula for $\int \sqrt{x^2 - a^2}\, dx$.

Solution

Let

$$x = a \sec t \qquad (0 \le t < \pi/2 \text{ or } \pi \le t < 3\pi/2)$$

Then $dx = a \sec t \tan t\, dt$ and (as already explained) $\sqrt{x^2 - a^2} = a \tan t$. Our integral becomes

$$\int \sqrt{x^2 - a^2}\, dx = \int a \tan t \cdot a \sec t \tan t\, dt = a^2 \int \sec t \tan^2 t\, dt$$

The device used in Example 2 (breaking off $\sec t \tan t$ in the hope of expressing the rest of the integrand in terms of $\sec t$) does not work this time. (Why?) Instead we write

$$\int \sec t \tan^2 t\, dt = \int \sec t\,(\sec^2 t - 1)\, dt = \int \sec^3 t\, dt - \int \sec t\, dt$$

The second integral on the right appears as a standard form in our table of integrals. The first is not easy, but it can be handled by integration by parts. In fact it has already appeared in Problem 17 of Section 10.1, the result of which is sufficiently useful to be recorded:

$$\int \sec^3 x\, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \ln|\sec x + \tan x| + C$$

Using these results, we find

$$\begin{aligned}
\int \sec t \tan^2 t\, dt &= \frac{1}{2}\sec t \tan t + \frac{1}{2}\ln|\sec t + \tan t| - \ln|\sec t + \tan t| + C \\
&= \frac{1}{2}\sec t \tan t - \frac{1}{2}\ln|\sec t + \tan t| + C \\
&= \frac{1}{2} \cdot \frac{x}{a} \cdot \frac{\sqrt{x^2 - a^2}}{a} - \frac{1}{2}\ln\left|\frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a}\right| + C \\
&= \frac{x}{2a^2}\sqrt{x^2 - a^2} - \frac{1}{2}\ln\left|\frac{x + \sqrt{x^2 - a^2}}{a}\right| + C \\
&= \frac{x}{2a^2}\sqrt{x^2 - a^2} - \frac{1}{2}\ln\left|x + \sqrt{x^2 - a^2}\right| + \frac{1}{2}\ln a + C
\end{aligned}$$

The constant $\frac{1}{2}\ln a$ may be dropped. (Why?) Hence the original integral is

$$\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ln\left|x + \sqrt{x^2 - a^2}\right| + C$$

■

We leave it to you (in the problem set) to derive the companion formula

$$\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

■ Example 4

The semicircular region $R = \{(x, y)\colon x^2 + y^2 \le a^2,\ y \ge 0\}$ is covered by a thin material whose density at (x, y) is $\delta(x, y) = y$. Find the center of mass of the material.

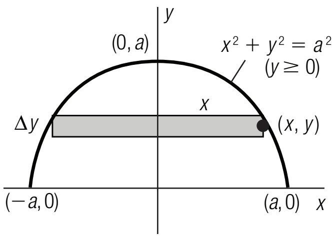

Figure 1 Center of mass by horizontal strips

Solution

See Figure 1, in which we have used a horizontal strip as the element of area (because the density is essentially constant in such a strip if it is thin). The mass of the strip is

$$\Delta m \approx \delta\,\Delta A \approx y \cdot 2x\,\Delta y = 2xy\,\Delta y$$

so the mass of the material covering R is

$$m = \int dm = 2\int_0^a y\sqrt{a^2 - y^2}\,dy$$

$$= -\int_{a^2}^{0} u^{1/2}\,du \qquad (u = a^2 - y^2,\ du = -2y\,dy)$$

$$= \frac{2}{3}u^{3/2}\Big|_0^{a^2} = \frac{2}{3}a^3$$

The moment of the strip relative to the x axis is

$$\Delta M_x \approx y\,\Delta m \approx 2xy^2\Delta y$$

so the total moment is

$$M_x = 2\int_0^a y^2\sqrt{a^2 - y^2}\,dy$$

This integral calls for the trigonometric substitution

$$y = a\sin t \qquad (-\pi/2 \le t \le \pi/2)$$

Since $dy = a\cos t\,dt$ and

$$\sqrt{a^2 - y^2} = \sqrt{a^2 - a^2\sin^2 t} = a\cos t$$

we find

$$M_x = 2\int_0^{\pi/2} a^2\sin^2 t \cdot a\cos t \cdot a\cos t\,dt = 2a^4\int_0^{\pi/2}\sin^2 t\cos^2 t\,dt$$

The device to be used at this point is not obvious, but you can see how it works. We use the identity

$$\sin 2t = 2\sin t\cos t \qquad (\text{from which } \sin^2 t\cos^2 t = \tfrac{1}{4}\sin^2 2t)$$

to write

$$M_x = \frac{a^4}{2}\int_0^{\pi/2}\sin^2 2t\,dt$$

Then the identity

$$\sin^2 t = \tfrac{1}{2}(1 - \cos 2t) \qquad (\text{with } t \text{ replaced by } 2t)$$

yields

$$M_x = \frac{a^4}{4}\int_0^{\pi/2}(1 - \cos 4t)\,dt = \frac{a^4}{4}\left(t - \frac{1}{4}\sin 4t\right)\Big|_0^{\pi/2} = \frac{\pi a^4}{8}$$

The y coordinate of the center of mass is therefore

$$\bar{y} = \frac{M_x}{m} = \frac{\pi a^4}{8} \cdot \frac{3}{2a^3} = \frac{3}{16}\pi a$$

The x coordinate is 0 because of the symmetry of the region. (Note that this is unaffected by the variable density, since $\delta = y$. The mass is distributed symmetrically about the y axis.) Thus the center of mass is

$$(\bar{x}, \bar{y}) = (0, \tfrac{3}{16}\pi a)$$

■

■ Example 5

To compute

$$\int_2^6 \frac{dx}{x^2\sqrt{4 + x^2}}$$

make the substitution

$$x = 2\tan t \qquad (-\pi/2 < t < \pi/2)$$

Then $dx = 2\sec^2 t\, dt$ and

$$\sqrt{4 + x^2} = \sqrt{4 + 4\tan^2 t} = 2\sec t$$

from which

$$\int \frac{dx}{x^2\sqrt{4 + x^2}} = \int \frac{2\sec^2 t\, dt}{4\tan^2 t \cdot 2\sec t} = \frac{1}{4}\int \frac{\sec t\, dt}{\tan^2 t}$$

Since there is no obvious way to work this out in terms of secant and tangent, we change to sine and cosine:

$$\int \frac{dx}{x^2\sqrt{4 + x^2}} = \frac{1}{4}\int \frac{\cos t\, dt}{\sin^2 t}$$

$$= \frac{1}{4}\int \frac{du}{u^2} \qquad (u = \sin t, du = \cos t\, dt)$$

$$= -\frac{1}{4}u^{-1} = -\frac{1}{4}\csc t \qquad \text{(arbitrary constant omitted)}$$

At this point we have two options. One is to change limits and write

$$\int_2^6 \frac{dx}{x^2\sqrt{4 + x^2}} = -\frac{1}{4}\csc t\,\Bigg|_{\pi/4}^{\tan^{-1}3} = \frac{1}{4}\left[\sqrt{2} - \csc(\tan^{-1}3)\right]$$

while the other is to return to the original variable (expressing $-\frac{1}{4}\csc t$ in terms of x).

The first option requires us to find $\csc(\tan^{-1}3)$. Letting $\theta = \tan^{-1}3$ (from which $\tan\theta = 3$), we observe from Figure 2 that

$$\csc(\tan^{-1}3) = \csc\theta = \frac{\sqrt{10}}{3}$$

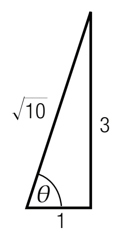

Figure 2 Finding $\csc(\tan^{-1}3)$

Hence

$$\int_2^6 \frac{dx}{x^2\sqrt{4 + x^2}} = \frac{1}{4}\left(\sqrt{2} - \frac{\sqrt{10}}{3}\right) = \frac{1}{12}(3\sqrt{2} - \sqrt{10})$$

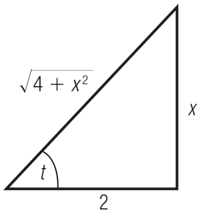

Figure 3 Finding csc t

The second option requires that we find csc t in terms of $x = 2\tan t$. Using Figure 3, we observe that $\csc t = \sqrt{4+x^2}/x$. Hence

$$\int_2^6 \frac{dx}{x^2\sqrt{4+x^2}} = -\frac{1}{4}\csc t\,\Big|_{x=2}^{x=6} = -\frac{1}{4}\cdot\frac{\sqrt{4+x^2}}{x}\,\Big|_2^6 = \frac{1}{4}\left(\frac{\sqrt{8}}{2} - \frac{\sqrt{40}}{6}\right)$$

$$= \frac{1}{4}\left(\sqrt{2} - \frac{\sqrt{10}}{3}\right) = \frac{1}{12}(3\sqrt{2} - \sqrt{10})$$

as before.

As you can see, there is not much to choose between these options. Note, however, the usefulness of the triangles. Drawing such figures is often helpful in problems involving trigonometric substitutions. ■

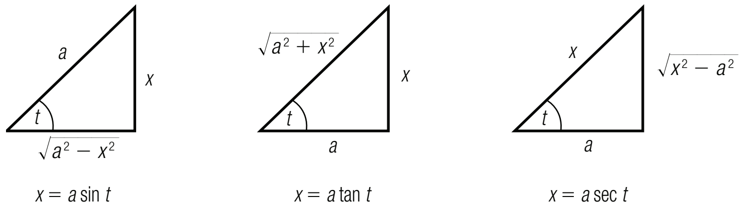

Figure 4 Triangles for trigonometric substitutions

Triangles that may be used in connection with our three trigonometric substitutions are shown in Figure 4. Because of our choice of domain for each (invertible) trigonometric function, these triangles apply when $x < 0$ as well as $x > 0$. They are drawn for the case $0 < t < \pi/2$ (t is an acute angle), in which case x is positive. Consider the third one, however, in the case $\pi < t < 3\pi/2$ (the other part of the range of $\sec^{-1}$). The proper way to draw it is shown in Figure 5. The legs are directed distances (both negative in this case), while the hypotenuse, as always, is an ordinary positive distance (because $x < 0$). These precautions are unnecessary, however, because the other functions come out just as they do in the acute case:

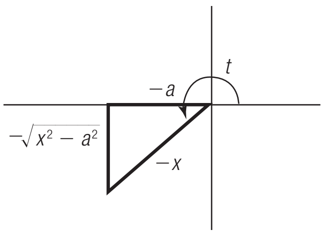

Figure 5 Triangle for $x = a\sec t$ $(\pi < t < 3\pi/2)$

$$\sin t = \frac{-\sqrt{x^2-a^2}}{-x} = \frac{\sqrt{x^2-a^2}}{x}$$

$$\cos t = \frac{-a}{-x} = \frac{a}{x}$$

$$\tan t = \frac{-\sqrt{x^2-a^2}}{-a} = \frac{\sqrt{x^2-a^2}}{a}$$

and so on.

It is worth noting that these trouble-free triangles are no accident! They are a consequence of our choice of domains in Section 9.1 (particularly in the case of $\sec^{-1}$, which as we noted at the time is not always defined the same way).

Problem Set 10.3

Find a formula for each of the following.

1. $\int \frac{dx}{x^2\sqrt{4-x^2}}$

2. $\int \frac{x^2\,dx}{\sqrt{25-x^2}}$

3. $\int \frac{\sqrt{1+x^2}}{x^2}\,dx$ *Hint:* $\int \frac{\sec^3 t}{\tan^2 t}\,dt$ can be written as a sum of manageable integrals by using $\sec^2 t = \tan^2 t + 1$.

4. $\int \frac{dx}{x^4\sqrt{x^2-1}}$ *Hint:* $\int \cos^3 t\,dt = \int (1-\sin^2 t)\cos t\,dt$

5. $\int \frac{dx}{(1-x^2)^{3/2}}$

6. $\int x^2\sqrt{4-x^2}\,dx$

7. $\int \frac{(1+x^2)^{3/2}\,dx}{x^6}$

8. $\int \frac{\sqrt{1-4x^2}}{x^2}\,dx$ *Hint:* $\int \cot^2 t\,dt = \int (\csc^2 t - 1)\,dt$

9. $\int \frac{x^2\,dx}{\sqrt{9+x^2}}$

10. $\int \frac{x^4\,dx}{\sqrt{1-x^2}}$

11. $\int x\sin^{-1} x\,dx$

12. $\int x\sinh^{-1} x\,dx$

Evaluate each of the following.

13. $\int_2^4 \frac{\sqrt{16-x^2}}{x^2}\,dx$

14. $\int_0^3 x^2\sqrt{9-x^2}\,dx$

15. $\int_0^2 \frac{x^2\,dx}{\sqrt{4+x^2}}$

16. $\int_0^1 \frac{x^2\,dx}{(4-x^2)^{3/2}}$

17. $\int_2^3 \frac{dx}{x^4\sqrt{16-x^2}}$

Hint: $\int \csc^4 t\,dt = \int (\cot^2 t + 1)\csc^2 t\,dt$

18. $\int_1^2 \frac{\sqrt{1+4x^2}}{x^4}\,dx$

19. Find $\int \frac{x\,dx}{x^2+9}$ in two ways, as follows.
    (a) Make an appropriate algebraic substitution.
    (b) Make a trigonometric substitution.
    (c) Reconcile the results.

20. Find
$$\int \frac{x\,dx}{\sqrt{x^2-1}}$$
in two ways, as follows.
    (a) Make an appropriate algebraic substitution.
    (b) Make a trigonometric substitution.
    (c) Reconcile the results.

21. Find
$$\int_0^3 \frac{x^3\,dx}{\sqrt{9+x^2}}$$
by making the substitution $u = \sqrt{9+x^2}$.
Hint: $u^2 = 9 + x^2$ and $u\,du = x\,dx$.

22. Find
$$\int_1^2 \frac{\sqrt{4-x^2}}{x}\,dx$$
in two ways, as follows.
    (a) Make the substitution $u = \sqrt{4-x^2}$.
    (b) Make a trigonometric substitution.

Problems 21 and 22 raise a question. If substitution of u for the radical works in these problems, why not in all the earlier ones? The answer is that $x\,dx$ is called for in the integrand; only if there is an *odd* power of x already present (becoming *even* when the needed x is supplied) is the substitution effective.

23. Find the length of the parabolic arc $y = \frac{1}{2}x^2$, $0 \le x \le 2$.

24. Find the length of the curve $y = \ln x$, $1 \le x \le 2$.

25. Solve the initial value problem
$$\frac{dy}{dx} = \frac{\sqrt{x^2-1}}{x^2} \qquad y = 0 \text{ when } x = 1$$

Each of the following standard forms has already been derived. Use a trigonometric substitution to confirm it. (Note the comparative simplicity of the argument, with the possible exception of the last formula, which requires some manipulation.)

26. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$

27. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$

28. $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x + \sqrt{x^2-a^2}\right| + C$

29. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$

30. Derive the formula

$$\int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

31. The triangles in Figure 4 are drawn for the case $0 < t < \pi/2$, when

$$x = a\sin t \qquad x = a\tan t \quad \text{or} \quad x = a\sec t$$

is positive. In Figure 5 we confirmed that the third one works out when x is negative. Do the same thing with the first and second triangles.

Hyperbolic substitutions work as well as trigonometric ones, the techniques being much the same. The following problems suggest the possibilities (and connections with earlier results).

32. Derive the formula

$$\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\frac{x}{a} + C$$

by substituting $x = a\sinh t$ and using the identity $\cosh^2 t - \sinh^2 t = 1$.

33. Derive the formula

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

by substituting $x =$ a $\sinh t$ and using the standard form

$$\int \operatorname{sech} t\,dt = \tan^{-1}(\sinh t) + C$$

34. Derive the formula

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C$$

as follows.

(a) Let $x = a\tanh t$ and use the identity $\tanh^2 t + \operatorname{sech}^2 t = 1$ (together with the integral of sech) to obtain

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \tan^{-1}(\sinh t) + C$$

(b) Use the identity

$$\tan^{-1}(\sinh t) = \sin^{-1}(\tanh t)$$

given in Additional Problem 45, Chapter 9.

35. Use the substitution $x = 2\sinh t$ to find

$$\int_2^6 \frac{dx}{x^2\sqrt{4 + x^2}}$$

and compare with Example 5.

36. Show that if $x = 2\tan^{-1} u$, then

$$dx = \frac{2\,du}{1 + u^2} \qquad \sin x = \frac{2u}{1 + u^2} \qquad \cos x = \frac{1 - u^2}{1 + u^2}$$

Use the substitution in Problem 36 to find the following integrals.

37. $\displaystyle\int \frac{dx}{1 + \cos x}$

38. $\displaystyle\int_0^{\pi/3} \frac{dx}{1 - \sin x}$

39. $\displaystyle\int \frac{dx}{\sin x + \tan x}$

40. $\displaystyle\int \frac{dx}{\tan x - \sin x}$

10.4 Decomposition of Rational Functions into Partial Fractions

The idea of this section is that *rational functions* (quotients of polynomials) can be integrated by breaking them into simpler parts. We begin with an example that leads to such an integration problem.

■ Example 1

Suppose that the earth can support no more than 10 billion people. The *Law of Inhibited Growth* is a model of the situation in which we assume that the population x grows at a rate jointly proportional to x itself and $10 - x$ (the difference between x and its upper limit, measured in billions). In other words,

$$\frac{dx}{dt} = kx(10 - x) \qquad \text{(where } t \text{ is time and } k \text{ is a constant)}$$

To solve this differential equation, we separate the variables and integrate:

$$\frac{dx}{x(10-x)} = k\,dt$$

$$\int \frac{dx}{x(10-x)} = \int k\,dt = kt + C$$

The integral on the left side is of the type we propose to discuss (because the integrand is a rational function). The idea is to "decompose" the function into a sum of manageable fractions (called "partial fractions"). When this has been done, the population may be found as a function of time by carrying out the integration and solving for x. (See the problem set. For an interesting account of research involving such analysis, see *Human Population Growth: Stability or Catastrophe*? by David A. Smith, in the September 1977 issue of *Mathematics Magazine.*) ■

In Example 1 it is not hard to see which decomposition we should try:

$$\frac{1}{x(10-x)} = \frac{A}{x} + \frac{B}{10-x} \qquad (1)$$

where A and B are constants to be specified later. We expect such a decomposition to work because the denominator $x(10-x)$ on the left side is the common denominator of the fractions on the right. When these fractions are recombined, we should be able to make the numerators on each side the same by an appropriate choice of A and B.

To see what this choice is, multiply each side of Equation (1) by $x(10-x)$, obtaining the identity

$$1 = A(10-x) + Bx \qquad (2)$$

This equation is true for all values of x, even $x = 0$ and $x = 10$. [These values are excluded in Equation (1), but they are legitimate in Equation (2) because both sides are continuous.] *Hence we may substitute any values of x we choose in Equation* (2). Two substitutions will enable us to evaluate the constants A and B. (Why?)

The most convenient substitutions in Equation (2) are $x = 0$ and $x = 10$, because they cause one term or the other (on the right side) to drop out. Letting $x = 0$, we find $1 = 10A$ (from which $A = \frac{1}{10}$), while substitution of $x = 10$ yields $1 = 10B$ (and hence $B = \frac{1}{10}$).

Thus we have discovered that

$$\frac{1}{x(10-x)} = \frac{1}{10}\cdot\frac{1}{x} + \frac{1}{10}\cdot\frac{1}{10-x}$$

and our integration problem in Example 1 is solved:

$$\begin{aligned}\int \frac{dx}{x(10-x)} &= \frac{1}{10}\int\frac{dx}{x} + \frac{1}{10}\int\frac{dx}{10-x}\\ &= \frac{1}{10}\ln x - \frac{1}{10}\ln(10-x) \quad \text{(arbitrary constant omitted)}\\ &= \frac{1}{10}\ln\frac{x}{10-x}\end{aligned}$$

(We left out the usual absolute values because the population limitations in Example 1 put x between 0 and 10. Hence x and $10 - x$ are positive.)

The substance of this section is that any rational function can be integrated as in the preceding example (by decomposition followed by use of standard formulas already developed). Rather than dwelling on the theorems from algebra that justify this statement, we present a number of examples to show you what we mean. The algebraic theory will emerge as we proceed.

■ Example 2

Find

$$\int \frac{x^3 + 2}{x^2 - x - 6}\,dx$$

Solution

The integrand is an *improper* rational fraction, meaning that the degree of the numerator is at least as large as the degree of the denominator. In such circumstances, long division reduces the fraction to *the sum of a polynomial and a proper fraction:*

$$\frac{x^3 + 2}{x^2 - x - 6} = (x + 1) + \frac{7x + 8}{x^2 - x - 6}$$

(Confirm!) Since we can certainly integrate a polynomial, our discussion of decomposition may be confined to proper fractions.

Factoring the denominator of our proper fraction, we have

$$\frac{7x + 8}{x^2 - x - 6} = \frac{7x + 8}{(x - 3)(x + 2)}$$

If this is going to break up into a sum, the only (simple) possibility is

$$\frac{7x + 8}{(x - 3)(x + 2)} = \frac{A}{x - 3} + \frac{B}{x + 2}$$

(for reasons explained in the discussion following Example 1). Since this equation implies that

$$7x + 8 = A(x + 2) + B(x - 3) \qquad \text{for all } x$$

we may substitute any values of x that seem convenient. Putting $x = 3$, we find $29 = 5A$ (or $A = \frac{29}{5}$), while $x = -2$ gives $-6 = -5B$ (or $B = \frac{6}{5}$). Thus

$$\frac{7x + 8}{(x - 3)(x + 2)} = \frac{29}{5} \cdot \frac{1}{x - 3} + \frac{6}{5} \cdot \frac{1}{x + 2}$$

and hence

$$\int \frac{x^3 + 2}{x^2 - x - 6}\,dx = \int (x + 1)\,dx + \frac{29}{5}\int \frac{dx}{x - 3} + \frac{6}{5}\int \frac{dx}{x + 2}$$

$$= \frac{1}{2}x^2 + x + \frac{29}{5}\ln|x - 3| + \frac{6}{5}\ln|x + 2| + C$$ ■

Remark

The substitution of $x = 3$ and $x = -2$ in Example 2 bypasses a longer method that is sometimes needed. The identity

$$7x + 8 = A(x + 2) + B(x - 3)$$

can be written in the form

$$7x + 8 = (A + B)x + (2A - 3B)$$

Equal polynomials have equal coefficients (of corresponding terms), which implies in this case that

$$A + B = 7 \qquad \text{and} \qquad 2A - 3B = 8$$

This is a system of linear equations in A and B, yielding $A = \frac{29}{5}$ and $B = \frac{6}{5}$ (as you can check). However, you should avoid this approach (or at least cut it down) whenever the substitution of special values of x promises to yield one or more of the unknown constants.

Example 2 works out nicely because the denominator $x^2 - x - 6$ is easily factored into $(x - 3)(x + 2)$. All the examples and problems in this section are chosen to avoid difficult factoring, but nevertheless you should be aware of a general theorem about the question. It is proved in algebra that *any polynomial with real coefficients can be expressed as a product of real factors of degree no higher than 2*. For example,

$$x^2 - x - 6 = (x - 3)(x + 2)$$

(a product of linear factors alone), while

$$x^4 - 16 = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4)$$

(a product of both linear and quadratic factors). The point of this theorem (in the present context) is that the *denominator* of any rational function we encounter may always be factored this way. The quadratic factors (if any) are understood to be *irreducible* (that is, they cannot be factored further using real coefficients). For example, $x^2 - 4$ is *reducible* because it factors into $(x - 2)(x + 2)$, but $x^2 + 4$ is irreducible.

■ Example 3

Find

$$\int \frac{x + 5}{x^3 - 2x^2 + x}\,dx$$

Solution

The integrand is already proper, so we factor its denominator:

$$x^3 - 2x^2 + x = x(x^2 - 2x + 1) = x(x - 1)^2$$

Thus the problem is to decompose

$$\frac{x + 5}{x(x - 1)^2}$$

It is not so obvious this time what form the decomposition should take. A little thought, however, should convince you that partial fractions of the type

$$\frac{A}{x} \qquad \frac{B}{x - 1} \qquad \frac{C}{(x - 1)^2}$$

may all be involved in a sum that gives back the original fraction when its terms are recombined. Moreover, no other fractions *need* be involved. Hence our decomposition has the form

$$\frac{x+5}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

Since the least common denominator of the fractions on the right side is $x(x-1)^2$, we arrive at the equation

$$x + 5 = A(x-1)^2 + Bx(x-1) + Cx \qquad \text{for all } x$$

The useful special values of x this time are $x = 0$ and $x = 1$, yielding $A = 5$ and $C = 6$ (respectively) when they are substituted. To find B, rewrite the identity in the form

$$\begin{aligned} x + 5 &= A(x^2 - 2x + 1) + B(x^2 - x) + Cx \\ &= (A + B)x^2 + (-2A - B + C)x + A \end{aligned}$$

and equate the coefficients of x^2 on each side:

$$0 = A + B \qquad (\text{from which } B = -5 \text{ because } A = 5)$$

(In reasonably simple cases, like this one, you can probably put down $0 = A + B$ by inspection without rewriting the identity.)

Thus we have found the decomposition

$$\frac{x+5}{x(x-1)^2} = \frac{5}{x} - \frac{5}{x-1} + \frac{6}{(x-1)^2}$$

and our integral is

$$\begin{aligned} \int \frac{x+5}{x^3 - 2x^2 + x}\,dx &= 5\int \frac{dx}{x} - 5\int \frac{dx}{x-1} + 6\int \frac{dx}{(x-1)^2} \\ &= 5\ln|x| - 5\ln|x-1| - \frac{6}{x-1} + C \\ &= 5\ln\left|\frac{x}{x-1}\right| - \frac{6}{x-1} + C \end{aligned}$$

■

Examples 2 and 3 suggest what happens in general when the denominator of our fraction involves linear factors:

- A linear factor $ax + b$ that occurs only once calls for a term of the form $A/(ax + b)$ in the decomposition.
- A repeated linear factor, say $(ax + b)^k$, calls for a sum of terms of the form

$$\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \ldots + \frac{A_k}{(ax+b)^k}$$

(The second statement includes the first as a special case by taking $k = 1$.)

Now we turn to problems in which (irreducible) quadratic factors are allowed as well.

■ Example 4

Find

$$\int \frac{x^2 - x + 5}{x(x^2 + 1)}\,dx$$

Solution

The integrand is proper and the denominator is factored, so the preliminaries are already out of the way. The decomposition takes the form

$$\frac{x^2 - x + 5}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$

Note the *linear* (not constant) numerator in the second fraction. If we wrote only

$$\frac{A}{x} + \frac{C}{x^2 + 1} \qquad \left(\text{leaving out } \frac{Bx}{x^2 + 1}\right)$$

we would not be putting down all the fractions that might contribute.

Using the least common denominator as before, we find

$$x^2 - x + 5 = A(x^2 + 1) + x(Bx + C) \qquad \text{for all } x$$

The only special value of x of any interest this time is $x = 0$, substitution of which gives $A = 5$. That leaves two unknowns; we need two equations to determine their values. Hence we equate coefficients of both x^2 and x:

$$1 = A + B \qquad \text{and} \qquad -1 = C$$

Since $A = 5$, we find $B = -4$ and our decomposition reads

$$\frac{x^2 - x + 5}{x(x^2 + 1)} = \frac{5}{x} + \frac{(-4x - 1)}{x^2 + 1} = \frac{5}{x} - \frac{4x}{x^2 + 1} - \frac{1}{x^2 + 1}$$

The integral is

$$\int \frac{x^2 - x + 5}{x(x^2 + 1)}\,dx = 5\int \frac{dx}{x} - 4\int \frac{x\,dx}{x^2 + 1} - \int \frac{dx}{x^2 + 1}$$

$$= 5 \ln|x| - 2 \ln(x^2 + 1) - \tan^{-1} x + C$$

■ Example 5

Find

$$\int \frac{2x^2 + x + 7}{(x^2 + 4)^2}\,dx$$

Solution

The repeated quadratic factor calls for a decomposition of the form

$$\frac{2x^2 + x + 7}{(x^2 + 4)^2} = \frac{Ax + B}{x^2 + 4} + \frac{Cx + D}{(x^2 + 4)^2}$$

(as in the linear case). We find

$$2x^2 + x + 7 = (x^2 + 4)(Ax + B) + (Cx + D)$$

No special value of x is of any help this time, so we equate coefficients of corresponding powers of x.

$$x^3\colon 0 = A \qquad x^2\colon 2 = B \qquad x^1\colon 1 = 4A+C \qquad x^0\colon 7 = 4B + D$$

Using the values $A = 0$ and $B = 2$ in the last two equations, we find $C = 1$ and $D = -1$. Our decomposition is

$$\frac{2x^2 + x + 7}{(x^2+4)^2} = \frac{2}{x^2+4} + \frac{x-1}{(x^2+4)^2} = \frac{2}{x^2+4} + \frac{x}{(x^2+4)^2} - \frac{1}{(x^2+4)^2}$$

and our integral is

$$\int \frac{2x^2 + x + 7}{(x^2+4)^2}\,dx = 2\int \frac{dx}{x^2+4} + \int \frac{x\,dx}{(x^2+4)^2} - \int \frac{dx}{(x^2+4)^2}$$

$$= \tan^{-1}\frac{x}{2} - \frac{1}{2(x^2+4)} - ?$$

The question mark is inserted because the third integral is not easy enough to put down by inspection. (We hope the first two are!) Make the substitution

$$x = 2\tan t \qquad (-\pi/2 < t < \pi/2)$$

in this integral. Then $dx = 2\sec^2 t\,dt$ and $(x^2+4)^2 = 16\sec^4 t$, from which

$$\int \frac{dx}{(x^2+4)^2} = \int \frac{2\sec^2 t\,dt}{16\sec^4 t} = \frac{1}{8}\int \cos^2 t\,dt = \frac{1}{16}\int (1+\cos 2t)\,dt$$

$$= \frac{1}{16}\left(t + \frac{1}{2}\sin 2t\right) = \frac{1}{16}(t + \sin t\cos t)$$

$$= \frac{1}{16}\left(\tan^{-1}\frac{x}{2} + \frac{x}{\sqrt{x^2+4}}\cdot\frac{2}{\sqrt{x^2+4}}\right)$$

$$= \frac{1}{16}\tan^{-1}\frac{x}{2} + \frac{x}{8(x^2+4)} \qquad \text{(arbitrary constant omitted)}$$

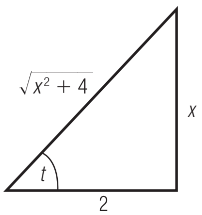

Figure 1 Finding $\sin t$ and $\cos t$

(See Figure 1.) Hence the original integral is

$$\int \frac{2x^2 + x + 7}{(x^2+4)^2}\,dx = \tan^{-1}\frac{x}{2} - \frac{1}{2(x^2+4)} - \frac{1}{16}\tan^{-1}\frac{x}{2} - \frac{x}{8(x^2+4)} + C$$

$$= \frac{15}{16}\tan^{-1}\frac{x}{2} - \frac{x+4}{8(x^2+4)} + C$$ ■

Examples 4 and 5 indicate that the rule for (irreducible) quadratic factors is the same as for linear factors, except that the numerators are linear instead of constant. None of these statements, incidentally, has been proved! We are relying on your intuition of what a reasonable decomposition should look like.

■ Example 6

Find

$$\int_2^3 \frac{dx}{x^3 - 1}$$

Solution

Since $x^3 - 1 = (x-1)(x^2+x+1)$ (the second factor being irreducible), we write

$$\frac{1}{x^3-1} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}$$

We leave it to you to confirm that $A = \frac{1}{3}$, $B = -\frac{1}{3}$, $C = -\frac{2}{3}$. Hence

$$\int \frac{dx}{x^3-1} = \frac{1}{3}\int \frac{dx}{x-1} - \frac{1}{3}\int \frac{(x+2)\,dx}{x^2+x+1} = \frac{1}{3}\ln|x-1| - \frac{1}{3}(?)$$

To find

$$\int \frac{(x+2)\,dx}{x^2+x+1}$$

let $u = x^2+x+1$, $du = (2x+1)\,dx$. Since the numerator is $x+2$ (when we would like it to be $2x+1$), we fix it up by writing

$$x + 2 = \frac{1}{2}(2x+1) + \frac{3}{2}$$

Then

$$\int \frac{(x+2)\,dx}{x^2+x+1} = \frac{1}{2}\int \frac{(2x+1)\,dx}{x^2+x+1} + \frac{3}{2}\int \frac{dx}{x^2+x+1}$$

$$= \frac{1}{2}\int \frac{du}{u} + \frac{3}{2}\int \frac{dx}{x^2+x+1}$$

The first integral on the right is a logarithm. The second is an inverse tangent, as you can see by completing the square in the denominator:

$$x^2 + x + 1 = (x^2 + x + \tfrac{1}{4}) + \tfrac{3}{4} = (x + \tfrac{1}{2})^2 + \tfrac{3}{4}$$

This prepares us for integration using the standard form

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Hence

$$\int \frac{(x+2)\,dx}{x^2+x+1} = \frac{1}{2}\int \frac{du}{u} + \frac{3}{2}\int \frac{dx}{(x+\frac{1}{2})^2 + \frac{3}{4}}$$

$$= \frac{1}{2}\ln|u| + \frac{3}{2}\cdot\frac{1}{\sqrt{3}/2}\tan^{-1}\left(\frac{x+1/2}{\sqrt{3}/2}\right)$$

$$= \frac{1}{2}\ln(x^2+x+1) + \sqrt{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

Thus

$$\int_2^3 \frac{dx}{x^3-1} = \left[\frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2+x+1) - \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)\right]\Bigg|_2^3$$

$$= \frac{1}{3}\ln 2 - \frac{1}{6}\ln 13 + \frac{1}{6}\ln 7 - \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{7}{\sqrt{3}}\right) + \frac{\sqrt{3}}{3}\tan^{-1}\left(\frac{5}{\sqrt{3}}\right)$$

≈ 0.075 (from a calculator) ■

Problem Set 10.4

Use the methods of this section to find each of the following.

1. $\int \frac{dx}{x^2+2x-3}$

2. $\int \frac{x\,dx}{x^2-5x+6}$

3. $\int \frac{x^2+1}{x(x^2-1)}\,dx$

4. $\int \frac{(x-3)\,dx}{x(x^2+x-2)}$

5. $\int \frac{dx}{x(x-1)^2}$

6. $\int \frac{x^3+1}{x^2(x-1)}\,dx$

7. $\int \frac{dx}{x(x^2+4)}$

8. $\int \frac{x^2\,dx}{(x-1)(x^2+1)}$

9. $\int \frac{x^2}{x^3-8}\,dx$ *Hint:* See the third quotation at the beginning of this chapter.

10. $\int \frac{dx}{x^4-1}$

11. $\int \frac{x^3\,dx}{(x^2+1)^2}$

12. $\int \frac{x^2-x+1}{(x^2+4)^2}\,dx$

13. $\int \frac{dx}{x^3-8}$

14. $\int \frac{dx}{x^3-27}$

Evaluate each of the following.

15. $\int_1^2 \frac{dx}{x^3(x+2)}$

16. $\int_3^5 \frac{x^3}{x^4+x^2-20}\,dx$

17. $\int_2^3 \frac{dx}{x^3(x^2-2x+1)}$

18. $\int_{-2}^2 \frac{x^4}{(x^2+4)^2}\,dx$

19. $\int_0^1 \frac{dx}{x^3+1}$

20. $\int_0^2 \frac{dx}{x^3+8}$

21. $\int_{-1}^0 \frac{x}{x^3-8}\,dx$

22. $\int_0^1 x^3 \tan^{-1} x\,dx$

23. $\int_0^{\pi/4} \frac{dx}{\tan x+1}$ *Hint:* Let $u=\tan x$.

24. To find

$$\int \frac{x^3+1}{x(x^2+1)}\,dx$$

suppose we write

$$\frac{x^3+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

(a) Show that this implies $A=1$, $B=-1$, $C=0$ and hence

$$\frac{x^3+1}{x(x^2+1)} = \frac{1}{x} - \frac{x}{x^2+1}$$

(b) When $x=1$ the "identity" in part (a) yields $1=\frac{1}{2}$. What went wrong?

(c) Find the integral.

25. To find

$$\int \frac{dx}{(x^2+1)^2}$$

suppose we write

$$\frac{1}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

(a) What is the result and how could it have been foreseen?

(b) Find the integral.

26. Find

$$\int \frac{dx}{x(10-x)}$$

by completing the square in the denominator and using a standard integration formula. (Assume that $0<x<10$, as in the opening example of this section.)

27. Find

$$\int \frac{dx}{x^2+2x-3}$$

by completing the square. (Compare with Problem 1.)

28. Use the ideas of this section to derive the standard form

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

29. Derive the formula

$$\int \frac{dx}{x(ax+b)} = \frac{1}{b}\ln\left|\frac{x}{ax+b}\right| + C$$

30. Derive the formula

$$\int \frac{dx}{x^2(ax+b)} = -\frac{1}{bx} + \frac{a}{b^2}\ln\left|\frac{ax+b}{x}\right| + C$$

31. Derive the formula

$$\int \frac{x}{(ax+b)^2}\,dx = \frac{1}{a^2}\left(\ln|ax+b| + \frac{b}{ax+b}\right) + C$$

32. Find the area of the region bounded by the curve $y=(x^2-1)/(x^2+1)$ and the x axis.

33. Find the centroid of the region in Problem 32.

34. Find the area of the region bounded by the curves $y=x^2/(x^4-16)$, $x=1$, and the x axis.

35. Find the centroid of the region bounded by the curves $y=x/(x^2+1)$, $x=2$, and the x axis.

36. Find the volume of the solid generated when the region bounded by the curves $y = (x - 2)/[x(x - 4)]$, $x = 1$, and the x axis is rotated about the x axis.

37. The *Law of Mass Action* in chemistry leads to the differential equation

$$\frac{dx}{dt} = k(a - x)(b - x) \qquad (k, a, b > 0)$$

where x is the amount at time t of a substance being formed from the reaction of two others ($x = 0$ at $t = 0$).

(a) Assuming that $a \neq b$, solve this initial value problem to obtain

$$\frac{a - x}{b - x} = \frac{a}{b} e^{(a - b)kt}$$

(b) Explain why x approaches the smaller of a and b as time goes on.

(c) Taking $a = 3$ and $b = 6$, suppose that 1 gram of the substance is formed in 10 minutes. How many grams are present 10 minutes later?

(d) Solve the differential equation in the case $a = b$.

38. Suppose that the upper limit of world population is 10 billion and that there were 2 billion people in 1920 and 6 billion in 2000.

(a) Use the Law of Inhibited Growth to show that the population t years after 1920 is

$$x = \frac{10}{1 + 4e^{-10kt}} \qquad \text{where } k = \frac{1}{800} \ln 6$$

What does x approach as time goes on?

(b) When will the population be 8 billion?

(c) If the Law of Exponential Growth ($x = x_0 e^{ct}$) is used instead, when will the population be 8 billion?

39. Suppose that the upper limit of world population is 8 billion, and assume the Law of Inhibited Growth (together with the statistics of 2 billion people in 1920 and 6 billion in 2000).

(a) Show that the population t years after 1920 is

$$x = \frac{8}{1 + 3e^{-8kt}} \qquad \text{where } k = \frac{1}{320} \ln 3$$

What does x approach as time goes on?

(b) When will the population be 7 billion?

(c) If the Law of Exponential Growth ($x = x_0 e^{ct}$) is used instead, when will the population be 7 billion?

40. A law of population growth that applies in some circumstances is

$$\frac{dx}{dt} = kx + ax^2 \qquad (\text{where } k \text{ and } a \text{ are positive})$$

(a) Solve this differential equation to obtain

$$x = \frac{k}{(a + k/x_0)e^{-kt} - a}$$

where x_0 is the initial population. What would this become if a were 0? *Hint:* Use Problem 29.

(b) Explain why it is unreasonable to use this model of population growth over a long period of time.

(c) More precisely, show that the population grows without bound in a finite time. When is doomsday?

41. In a town of N people a certain disease is spreading at a rate proportional to the product of the number already infected and the number not yet infected:

$$\frac{dy}{dt} = ky(N - y)$$

where y is the number infected at time t

Find y as a function of t if $y = 1$ when $t = 0$. What does y approach as $t \to \infty$?

42. Use the substitution $x = 2 \tan^{-1} u$ (Problem 36, Section 10.3) to find

$$\int \frac{dx}{1 + \sin x - \cos x}$$

(This substitution changes rational functions of $\sin x$ and $\cos x$ to rational functions of u.)

43. Repeat Problem 42 in the case of

$$\int \frac{dx}{1 + \cos x - \sin x}$$

10.5 Miscellaneous Integration Problems

As we mentioned at the beginning of this chapter, there are only two general methods of integration. One is *substitution* (including many special devices not covered in this book); the other is *integration by parts.* (Decomposition of rational functions is not really a method of integration, but an algebraic technique for breaking up a fraction into a sum.)

Nothing new is offered in this section (except for fuller explanation of some substitutions that have previously occurred only in the problem sets). Our purpose is to help you develop more confidence in your mastery of technique by presenting miscellaneous examples and problems.

■ Example 1

Find

$$\int \frac{\sqrt{x}\,dx}{x^2-1}$$

Solution

The substitution needed here was first suggested in Example 6, Section 6.5, but has not appeared too often since. Because the radical involves a *linear* expression, nothing so fancy as a trigonometric substitution is called for; simply let $u = \sqrt{x}$, $u^2 = x$, $2u\,du = dx$. Then

$$\int \frac{\sqrt{x}\,dx}{x^2-1} = \int \frac{u(2u\,du)}{u^4-1} = \int \frac{2u^2\,du}{u^4-1}$$

an integral that calls for a decomposition of the type described in the last section. We leave it to you to confirm that the integrand is

$$\frac{2u^2}{(u-1)(u+1)(u^2+1)} = \frac{1}{2}\cdot\frac{1}{u-1} - \frac{1}{2}\cdot\frac{1}{u+1} + \frac{1}{u^2+1}$$

Hence

$$\int \frac{\sqrt{x}\,dx}{x^2-1} = \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| + \tan^{-1}u + C$$

$$= \frac{1}{2}\ln\left|\frac{\sqrt{x}-1}{\sqrt{x}+1}\right| + \tan^{-1}\sqrt{x} + C$$ ■

■ Example 2

Find

$$\int_0^1 x^2 \sin^{-1} x\,dx$$

Solution

Integration by parts seems a good way to start. Let

$$u = \sin^{-1} x \qquad \text{and} \qquad dv = x^2\,dx$$

Then $du = dx/\sqrt{1-x^2}$ and $v = x^3/3$, from which

$$\int x^2 \sin^{-1} x\,dx = \frac{x^3}{3}\sin^{-1} x - \frac{1}{3}\int \frac{x^3\,dx}{\sqrt{1-x^2}}$$

The new integral can be handled by a trigonometric substitution. In view of the odd power in the numerator, however (see Problems 21 and 22, Section 10.3), we will let

$$u = \sqrt{1-x^2} \qquad u^2 = 1 - x^2 \qquad u\,du = -x\,dx$$

Then

$$\int \frac{x^3\,dx}{\sqrt{1-x^2}} = \int \frac{(1-u^2)(-u\,du)}{u}$$

$$= \int (u^2-1)\,du = \frac{u^3}{3} - u = \frac{u}{3}(u^2-3)$$

(We omitted the arbitrary constant.) The original integral is

$$\int_0^1 x^2 \sin^{-1} x\,dx = \frac{x^3}{3}\sin^{-1} x\,\Big|_0^1 - \frac{u}{9}(u^2-3)\,\Big|_1^0 = \frac{\pi}{6} - \frac{2}{9}$$ ■

Remark

In Example 2 we delayed inserting the limits of integration until the end. A funny thing happens if they are inserted early:

$$\int_0^1 \sin^{-1} x\,dx = \frac{x^3}{3}\sin^{-1} x\,\Big|_0^1 - \frac{1}{3}\int_0^1 \frac{x^3\,dx}{\sqrt{1-x^2}}$$

The original integrand is continuous in [0,1], so the integral certainly exists. The new integral, however, is *improper* (see Problem 40, Section 9.2) because $x^3/\sqrt{1-x^2}$ is unbounded in the domain of integration. We cannot integrate from 0 to 1, but only from 0 to t (where t is close to, but less than, 1). The result is

$$\int_0^t \frac{x^3\,dx}{\sqrt{1-x^2}} = \frac{u}{3}(u^2-3)\,\Big|_1^{\sqrt{1-t^2}} \qquad (u = \sqrt{1-x^2})$$

$$= \frac{2}{3} - \frac{1}{3}\sqrt{1-t^2}\,(2+t^2)$$

which approaches $\frac{2}{3}$ as $t \to 1$. The improper integral may be assigned this value and all is well. Since we have not yet formally discussed the subject, however, this approach is a little tricky! In our first solution the difficulty evaporated when u cancelled in the step

$$\int \frac{(1-u^2)(-u\,du)}{u} = \int (u^2-1)\,du$$

It is easy to overlook improper integrals that arise in this way. (If you worked Problem 24 in Section 10.1, for example, you may have sailed right by one!)

■ Example 3

Find

$$\int_0^\pi \frac{\sin x\,dx}{4+\cos^2 x}$$

Solution

This becomes an inverse tangent upon substitution of $u = \cos x$, $du = -\sin x\,dx$:

$$\int_0^{\pi} \frac{\sin x\,dx}{4 + \cos^2 x} = -\int_1^{-1} \frac{du}{4 + u^2} = 2\int_0^1 \frac{du}{4 + u^2} \qquad \text{(why?)}$$

$$= \tan^{-1}\frac{u}{2}\Big|_0^1 = \tan^{-1}\frac{1}{2} \approx 0.46$$ ■

■ Example 4

Find

$$\int \frac{dx}{1 - \sin x + \cos x}$$

Solution

The integrand is a rational function of $\sin x$ and $\cos x$, which calls for a substitution we have mentioned only in the problem sets. (See Problems 36 through 40, Section 10.3, and Problems 42 and 43, Section 10.4.) It is not an obvious device, but one of those clever ideas that has been around for a long time (and is worth understanding). Let

$$x = 2\tan^{-1} u \qquad dx = \frac{2\,du}{1 + u^2}$$

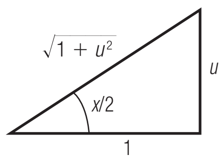

Figure 1 Finding $\sin x$ and $\cos x$

Since $\tan(x/2) = u$, we may use Figure 1 to find

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2 \cdot \frac{u}{\sqrt{1 + u^2}} \cdot \frac{1}{\sqrt{1 + u^2}} = \frac{2u}{1 + u^2}$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1}{1 + u^2} - \frac{u^2}{1 + u^2} = \frac{1 - u^2}{1 + u^2}$$

With these results in hand, we can write

$$\frac{dx}{1 - \sin x + \cos x} = \frac{1}{1 - \dfrac{2u}{1 + u^2} + \dfrac{1 - u^2}{1 + u^2}} \cdot \frac{2\,du}{1 + u^2} = \frac{du}{1 - u}$$

from which

$$\int \frac{dx}{1 - \sin x + \cos x} = \int \frac{du}{1 - u} = -\ln|1 - u| + C$$

$$= -\ln\left|1 - \tan\frac{x}{2}\right| + C$$ ■

The substitution in Example 4 is worth emphasizing:

> To find the integral of a rational function of $\sin x$ and $\cos x$, make the substitution
>
> $$x = 2\tan^{-1} u \qquad dx = \frac{2\,du}{1 + u^2}$$
>
> $$\sin x = \frac{2\,u}{1 + u^2} \qquad \cos x = \frac{1 - u^2}{1 + u^2}$$

■ Example 5

Find

$$\int_3^4 \frac{x\,dx}{x^4 - 16}$$

Solution

This appears to call for decomposition of the rational integrand. It is easier, however, to make the substitution $u = x^2$, $du = 2x\,dx$. Then

$$\int_3^4 \frac{x\,dx}{x^4 - 16} = \frac{1}{2}\int_9^{16} \frac{du}{u^2 - 16} = -\frac{1}{2}\int_9^{16} \frac{du}{16 - u^2} = -\frac{1}{2}\cdot\frac{1}{4}\coth^{-1}\frac{u}{4}\bigg|_9^{16}$$

$$= \frac{1}{8}\left(\coth^{-1}\frac{9}{4} - \coth^{-1} 4\right)$$

$$= \frac{1}{8}\left(\tanh^{-1}\frac{4}{9} - \tanh^{-1}\frac{1}{4}\right) \qquad [\text{because } \coth^{-1} x = \tanh^{-1}(1/x)]$$

$$\approx 0.028 \qquad (\text{from a calculator with a } \tanh^{-1} \text{ key})$$

The problem can also be done in terms of logarithms:

$$\int_3^4 \frac{x\,dx}{x^4 - 16} = \frac{1}{2}\int_9^{16} \frac{du}{u^2 - 16} = -\frac{1}{2}\int_9^{16} \frac{du}{16 - u^2} = -\frac{1}{16}\ln\left|\frac{4 + u}{4 - u}\right|\Bigg|_9^{16}$$

$$= -\frac{1}{16}\ln\frac{5}{3} + \frac{1}{16}\ln\frac{13}{5} = \frac{1}{16}\ln\frac{39}{25} \approx 0.028$$ ■

■ Example 6

Find

$$\int \frac{x\,dx}{\sqrt[3]{x + 1}}$$

Solution

As in Example 1, we substitute for the radical:

$$u = \sqrt[3]{x + 1} \qquad u^3 = x + 1 \qquad 3u^2\,du = dx$$

Then

$$\int \frac{x\,dx}{\sqrt[3]{x + 1}} = \int \frac{(u^3 - 1)(3u^2\,du)}{u}$$

$$= \int (3u^4 - 3u)\,du = \frac{3}{5}u^5 - \frac{3}{2}u^2 + C = \frac{3}{10}u^2(2u^3 - 5) + C$$

$$= \frac{3}{10}(x + 1)^{2/3}(2x - 3) + C$$ ■

■ Example 7

Find

$$\int \frac{x^4\,dx}{\sqrt{x^2-4}}$$

Solution

Make the trigonometric substitution

$$x = 2\sec t \qquad (0 \le t < \pi/2 \text{ or } \pi \le t < 3\pi/2)$$

Then $dx = 2\sec t\tan t\,dt$ and $\sqrt{x^2-4} = \sqrt{4\sec^2 t - 4} = 2\tan t$, from which

$$\int \frac{x^4\,dx}{\sqrt{x^2-4}} = \int \frac{16\sec^4 t \cdot 2\sec t\tan t\,dt}{2\tan t} = 16\int \sec^5 t\,dt$$

Now use integration by parts, letting $u = \sec^3 t$ and $dv = \sec^2 t\,dt$. Then

$$du = 3\sec^3 t\tan t\,dt \qquad \text{and} \qquad v = \tan t$$

from which

$$\begin{aligned}\int \sec^5 t\,dt &= \sec^3 t\tan t - 3\int \sec^3 t\tan^2 t\,dt \\ &= \sec^3 t\tan t - 3\int \sec^3 t\,(\sec^2 t - 1)\,dt \\ &= \sec^3 t\tan t - 3\int \sec^5 t\,dt + 3\int \sec^3 t\,dt\end{aligned}$$

Solving for $\int \sec^5 t\,dt$, and recalling from Section 10.3 that

$$\int \sec^3 t\,dt = \frac{1}{2}\sec t\tan t + \frac{1}{2}\ln|\sec t + \tan t| \qquad \text{(arbitrary constant omitted)}$$

we find

$$\begin{aligned}\int \frac{x^4\,dx}{\sqrt{x^2-4}} &= 16\left(\frac{1}{4}\sec^3 t\tan t + \frac{3}{4}\int \sec^3 t\,dt\right) \\ &= 4\sec^3 t\tan t + 6\sec t\tan t + 6\ln|\sec t + \tan t| + C \\ &= 2\sec t\tan t\,(2\sec^2 t + 3) + 6\ln|\sec t + \tan t| + C \\ &= 2\cdot\frac{x}{2}\cdot\frac{\sqrt{x^2-4}}{2}\left(2\cdot\frac{x^2}{4} + 3\right) + 6\ln\left|\frac{x}{2} + \frac{\sqrt{x^2-4}}{2}\right| + C \\ &= \frac{1}{4}x\sqrt{x^2-4}\,(x^2+6) + 6\ln\left|x + \sqrt{x^2-4}\right| + C\end{aligned}$$

(We dropped the constant term $-6\ln 2$.) ■

■ Example 8

Find

$$\int \frac{dx}{\sqrt{6x - x^2}}$$

Solution

We complete the square by writing

$$6x - x^2 = -(x^2 - 6x + 9) + 9 = 9 - (x - 3)^2$$

Then

$$\int \frac{dx}{\sqrt{6x - x^2}} = \int \frac{dx}{\sqrt{9 - (x - 3)^2}} = \sin^{-1}\left(\frac{x - 3}{3}\right) + C$$ ■

Problem Set 10.5

Find each of the following integrals.

1. $\int_1^3 \frac{dx}{x\sqrt{4 - x}}$

2. $\int \frac{dx}{x^2 + 8x + 20}$

3. $\int \sin \sqrt{x}\, dx$

4. $\int_0^1 \frac{e^x\, dx}{e^{2x} + 1}$

5. $\int_0^{\pi/6} \frac{\sin x \cos x}{1 - \sin x}\, dx$

6. $\int \frac{dx}{(a^2 - x^2)^{3/2}}$

7. $\int_0^1 \frac{x^2\, dx}{(x^2 + 1)^3}$

8. $\int_0^{\pi/4} \frac{\tan^2 x + 1}{\tan x + 1}\, dx$

9. $\int_0^{\pi/2} \frac{dx}{1 + \sin x}$

10. $\int \frac{x^4 + 1}{x^4 - 1}\, dx$

11. $\int \frac{dx}{1 + \sec x}$

12. $\int_0^1 \sqrt{4x - x^2}\, dx$

13. $\int_0^{\pi/4} \sec^4 x\, dx$

14. $\int_0^1 \frac{dx}{2 - \sqrt[3]{x}}$

15. $\int x \sec x \tan x\, dx$

16. $\int \sec^6 t\, dt$

17. $\int_0^1 \frac{x - 3}{x^2 - 2x - 8}\, dx$

18. $\int \frac{x^2\, dx}{\sqrt[3]{x + 2}}$

19. $\int_0^{\pi/4} \frac{\sec t}{1 + \sec^2 t}\, dt$

20. $\int \frac{dx}{x^2\sqrt{x^2 + 9}}$

21. $\int \frac{x^2 + 4}{x^3 - x}\, dx$

22. $\int_0^1 \sqrt{x^2 - x^4}\, dx$

23. $\int_0^1 x^3 \sin^{-1} x\, dx$

24. $\int \frac{e^t\, dt}{e^t - 1}$

25. $\int x^3 \sin x\, dx$

26. $\int_1^5 \frac{dx}{x\sqrt{x^2 + 10x}}$

27. $\int \frac{x\, dx}{\sqrt{6x - x^2}}$

28. $\int \frac{\sqrt{4 - x^2}}{x^3}\, dx$

29. $\int \frac{x\, dx}{x^4 + 1}$

30. $\int_0^3 \frac{(2x - 3)\, dx}{x^2 - 3x + 5}$

31. $\int e^x \cos 2x\, dx$

32. $\int_{\pi/4}^{\pi/3} \csc^4 t\, dt$

33. $\int \frac{dx}{x^2 - 4x}$

34. Find the area under the curve $y = (4 + x^2)/(4 - x^2)$, $-1 \le x \le 1$.

35. Find the length of the curve $y = x^2$, $0 \le x \le 1$.

36. Find the centroid of the region under the curve $y = e^x$, $0 \le x \le \ln 2$.

37. Find the volume of the solid generated when the region bounded by the curves $y = \cos x$, $x = \pm\pi/2$, and the x axis is rotated about the y axis.

38. Find the surface area generated when the region under the curve $y = 1/x$, $1 \le x \le 2$, is rotated about the x axis.

10.6 Numerical Integration

As we have pointed out before, a table of integrals (no matter how extensive) cannot touch some problems. A simple example is

$$\int_0^1 \sqrt{x^3 + 1}\, dx$$

which cannot be evaluated by any of the methods we have described. In fact no elementary function exists whose derivative is $\sqrt{x^3 + 1}$. Moreover, we tend to forget that our "evaluation" of many integrals is in name only. A result such as

$$\int_0^1 \cos x\, dx = \sin 1$$

has to be converted to a decimal approximation to be useful. While calculators give the values of common functions, there are situations in which we have to do better.

In this section we develop two formulas that enable us to compute

$$\int_a^b f(x)\, dx$$

as accurately as we please (provided of course that f itself is known to within the needed degree of precision). They are based on linear and quadratic approximation of $f(x)$, respectively.

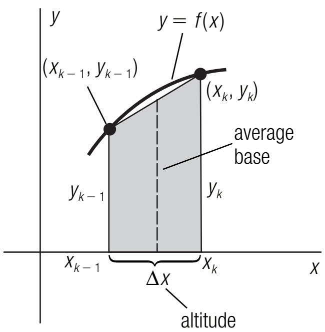

Figure 1 Trapezoidal approximation of area under a curve

Let $I = [a, b]$ and suppose that $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of I into n subintervals of equal length. In Figure 1 we show the graph of f in the typical subinterval, together with its linear approximation by a line segment joining its endpoints. The area of the shaded trapezoid (from geometry) is

$$\text{(average base)(altitude)} = \tfrac{1}{2}(y_{k-1} + y_k)\,\Delta x$$

To obtain an approximation of the integral of f from a to b, we need only add up these trapezoidal areas:

$$\int_a^b f(x)\, dx \approx \sum_{k=1}^{n} \frac{1}{2}(y_{k-1} + y_k)\,\Delta x$$

When this sum is written out, it becomes

$$\Delta x\left[\frac{1}{2}(y_0 + y_1) + \frac{1}{2}(y_1 + y_2) + \ldots + \frac{1}{2}(y_{n-2} + y_{n-1}) + \frac{1}{2}(y_{n-1} + y_n)\right]$$

$$= \Delta x\left(\frac{1}{2}y_0 + y_1 + y_2 + \ldots + y_{n-1} + \frac{1}{2}y_n\right)$$

Thus we have arrived at the following result.

The Trapezoidal Rule

Let $\{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ into subintervals of equal length $\Delta x = (b - a)/n$. If

$$y_k = f(x_k) \qquad k = 0, 1, \ldots, n$$

then

$$\int_a^b f(x)\, dx \approx \Delta x\left(\frac{1}{2}y_0 + y_1 + y_2 + \ldots + y_{n-1} + \frac{1}{2}y_n\right)$$

We do not claim to have proved anything! How could we, when no hypotheses are given concerning f and no precision is imposed on the approximation? Not until an upper bound is given for the error does the rule acquire any mathematical substance beyond the intuitive ideas of area that lead to it. Note, however, that if f is integrable (which of course we assume when we write its integral), the approximation does approach the right answer as n increases. To see why, let

$$\begin{aligned} T_n &= \Delta x\left(\frac{1}{2}y_0 + y_1 + y_2 + \dots + y_{n-1} + \frac{1}{2}y_n\right) \\ &= (y_1 + y_2 + \dots + y_n)\,\Delta x - \frac{1}{2}(y_n - y_0)\,\Delta x \\ &= \sum_{k=1}^{n} f(x_k)\Delta x - \frac{1}{2}\Big[f(b) - f(a)\Big]\Delta x \end{aligned}$$

When n increases (which forces $\Delta x \to 0$), we find

$$\lim_{\Delta x\to 0} T_n = \lim_{\Delta x\to 0} \sum_{k=1}^{n} f(x_k)\Delta x - 0 = \int_a^b f(x)\,dx$$

This guarantees that the Trapezoidal Rule approximates the integral as closely as we please (when n is taken sufficiently large). More precisely, it can be proved that if f'' exists in I, and M is an upper bound for $|f''(x)|$ in I, that is,

$$|f''(x)| \le M \qquad \text{for } a \le x \le b$$

then the error in the Trapezoidal Rule is

$$E_n = \left|\int_a^b f - T_n\right| \le \frac{1}{12}(b-a)\,M\,(\Delta x)^2$$

■ Example 1

Use the Trapezoidal Rule with $n = 4$ to estimate

$$\int_1^3 \frac{dx}{x}$$

Solution

Since $\Delta x = \frac{1}{2}$, the points of subdivision are $x_0 = 1$, $x_1 = \frac{3}{2}$, $x_2 = 2$, $x_3 = \frac{5}{2}$, $x_4 = 3$. The corresponding functional values $(y_k = 1/x_k)$ are $y_0 = 1$, $y_1 = \frac{2}{3}$, $y_2 = \frac{1}{2}$, $y_3 = \frac{2}{5}$, $y_4 = \frac{1}{3}$. Hence

$$\int_1^3 \frac{dx}{x} \approx \frac{1}{2}\left(\frac{1}{2} + \frac{2}{3} + \frac{1}{2} + \frac{2}{5} + \frac{1}{6}\right) \approx 1.12$$

Since

$$\int_1^3 \frac{dx}{x} = \ln 3 \approx 1.10$$

our approximation is correct in the first decimal place. ■

■ Example 2

Use the Trapezoidal Rule with $n = 4$ to estimate

$$\int_0^1 \sqrt{x^3 + 1}\, dx$$

Solution

Since $\Delta x = 0.25$, the points of subdivision are $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$. The corresponding values of the integrand (rounded off from a calculator) are $y_0 = 1$, $y_1 = 1.0078$, $y_2 = 1.0607$, $y_3 = 1.1924$, $y_4 = 1.4142$. Hence

$$\int_0^1 \sqrt{x^3 + 1}\, dx \approx 0.25(0.5 + 1.0078 + 1.0607 + 1.1924 + 0.7071) = 1.117$$

This time (unlike Example 1) we have no answer to serve as a check. We can, however, look at the error. If $f(x) = \sqrt{x^3 + 1}$, then

$$f'(x) = \frac{3x^2}{2\sqrt{x^3 + 1}} \qquad \text{and} \qquad f''(x) = \frac{3x(x^2 + 4)}{4(x^3 + 1)^{3/2}}$$

An upper bound of $|f''(x)|$ for $0 \le x \le 1$ is $M = \frac{15}{4}$ (why?), so the error is

$$E_4 \le \tfrac{1}{12}(b - a)M(\Delta x)^2 = \tfrac{1}{12}(1)(\tfrac{15}{4})(0.25)^2 < 0.02$$

Our approximation is therefore accurate to at least one decimal place. ■

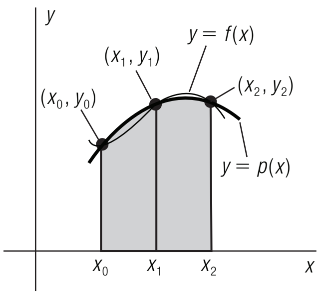

Figure 2 Parabolic approximation to a curve

Now we turn to a numerical integration formula based on quadratic approximation of $f(x)$. Instead of joining consecutive points by straight line segments (as in Figure 1), we pass *parabolic arcs* through three points at a time. (Figure 2 shows the procedure in the first two subintervals of the partition.) In the problem set we ask you to prove that the area of the shaded region (under the parabola $y = p(x)$ which passes through the three points) is

$$\int_{x_0}^{x_2} p(x)\, dx = \frac{\Delta x}{3}(y_0 + 4y_1 + y_2)$$

Similarly, the area under the next parabola (from x_2 to x_4) is

$$\frac{\Delta x}{3}(y_2 + 4y_3 + y_4)$$

Assuming that n is even, we may continue in this way, the area under the last parabola (from x_{n-2} to x_n) being

$$\frac{\Delta x}{3}(y_{n-2} + 4y_{n-1} + y_n)$$

The sum of these areas is

$$\frac{\Delta x}{3}[(y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \ldots + (y_{n-2} + 4y_{n-1} + y_n)]$$

$$= \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n)$$

Note that with the exception of the first and last terms (which have coefficient 1) the coefficients are alternately 4 and 2. We summarize this result as follows.

Simpson's Parabolic Rule

Let $\{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$ into an even number of subintervals of equal length $\Delta x = (b - a)/n$. If

$$y_k = f(x_k) \qquad k = 0, 1, \ldots, n$$

then

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 2y_{n-2} + 4y_{n-1} + y_n)$$

Let S_n be the right side of Simpson's formula. It can be shown that if $f^{(4)}$ exists in I, and N is an upper bound of $|f^{(4)}(x)|$ for $a \le x \le b$, then the error is

$$E_n = \left| \int_a^b f - S_n \right| \le \frac{1}{180}(b - a)N(\Delta x)^4$$

Since $(\Delta x)^4$ approaches zero much faster than $(\Delta x)^2$ as $\Delta x \to 0$, we may reasonably expect that Simpson's Rule is more accurate than the Trapezoidal Rule. (This is not always the case, however, because M and N are not the same.)

■ Example 3

Use Simpson's Rule with $n = 4$ to estimate

$$\int_1^3 \frac{dx}{x}$$

Solution

As in Example 1, we have $\Delta x = \frac{1}{2}$ and $y_0 = 1$, $y_1 = \frac{2}{3}$, $y_2 = \frac{1}{2}$, $y_3 = \frac{2}{5}$, $y_4 = \frac{1}{3}$. Hence

$$\int_1^3 \frac{dx}{x} \approx \frac{1}{6}\left(1 + \frac{8}{3} + 1 + \frac{8}{5} + \frac{1}{3}\right) = 1.100$$

The true value of the integral is $\ln 3 = 1.0986\ldots$, so our approximation is accurate to two places. (Note the improvement over the Trapezoidal Rule.) ■

■ Example 4

Use Simpson's Rule with $n = 4$ to estimate

$$\int_0^1 \sqrt{x^3 + 1}\,dx$$

Solution

As in Example 2, we have $\Delta x = 0.25$ and $x_0 = 0$, $x_1 = 0.25$, $x_2 = 0.5$, $x_3 = 0.75$, $x_4 = 1$. Rounding off from a calculator, we find

$$\int_0^1 \sqrt{x^3 + 1}\,dx \approx \frac{0.25}{3}(1 + 4.0311 + 2.1213 + 4.7697 + 1.4142) = 1.1114$$

It is a nasty chore to estimate the error because we need the fourth derivative of $f(x) = \sqrt{x^3 + 1}$ and an upper bound for its absolute value in the interval [0,1]. In any case, however, the error is

$$E_4 \leq \frac{N}{180}\,(0.25)^4$$

The fourth power of 0.25 is considerably smaller than the second power (and of course $\frac{1}{180}$ is more than ten times smaller than $\frac{1}{12}$). This suggests that the error is less than in the Trapezoidal Rule, but we cannot be sure without including N. ■

All things considered, Simpson's Rule is superior to the Trapezoidal Rule (since parabolic arcs usually fit a curve better than line segments). It is no harder to compute, and the error is generally smaller. If you feel that the application of these rules is tedious, remember that to a computer they look easy. Computer-assisted numerical integration is the most practical way to find all but the simplest integrals.

Problem Set 10.6

Use the Trapezoidal Rule (with the given value of n) to compute an approximate value of each of the following integrals. When possible, compare with the true value of the integral.

1. $\displaystyle\int_1^2 \frac{dx}{x} \quad (n = 4)$

2. $\displaystyle\int_0^1 e^x\,dx \quad (n = 6)$

3. $\displaystyle\int_0^2 e^{-x}\,dx \quad (n = 6)$

4. $\displaystyle\int_0^{\pi/2} (1 - \cos x)\,dx \quad (n = 6)$

5. $\displaystyle\int_0^1 \frac{dx}{\sqrt{4 - x^2}} \quad (n = 4)$

6. $\displaystyle\int_0^1 \sqrt{1 - x^2}dx \quad (n = 4)$

7. $\displaystyle\int_0^{\pi} \frac{\sin x}{x}\,dx \quad (n = 4)$

   Hint: The integral is not improper. Define the integrand at $x = 0$ so as to make it continuous.

8. $\displaystyle\int_0^1 \sqrt{1 + x^4}\,dx \quad (n = 4)$

9. $\displaystyle\int_0^2 e^{-x^2}\,dx \quad (n = 4)$

10.–18. Use Simpson's Rule (with the given value of n) to compute an approximate value of each of the integrals in Problems 1 through 9. When possible, compare with the true value of the integral.

19.–22. Use the error bound given in the text to estimate your accuracy in Problems 1 through 4.

23.–26. Use the error bound given in the text to estimate your accuracy in Problems 10 through 13.

27. The width (at 2-inch intervals) of an irregularly shaped piece of material is shown in Figure 3. Use the Trapezoidal Rule to estimate the area of the material.

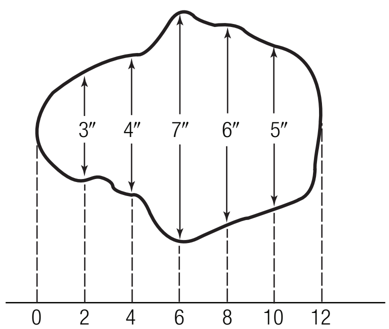

Figure 3 Irregular region

28. Repeat Problem 27 using Simpson's Rule.

29. The force applied to an object to move it 2 meters was measured at $\frac{1}{2}$-meter intervals (starting at the origin of the motion) and was found to be 15, 18, 20, 16, 18 newtons, respectively. Use the Trapezoidal Rule to estimate the work done.

30. Repeat Problem 29 using Simpson's Rule.

31. Use Simpson's Rule (with $n = 4$) to compute the length of the curve $y = \sin x$, $0 \leqslant x \leq \pi$.

32. Suppose that we want to compute

$$\int_1^2 \frac{dx}{x}$$

correct to five decimal places. If the Trapezoidal Rule is used, how many subintervals are needed to guarantee this accuracy? *Hint:* The error must be less than $\frac{1}{2} \times 10^{-5}$. (Why?)

33. Repeat Problem 32 using Simpson's Rule.

34. Suppose that $f(x)$ and $f''(x)$ are both positive in $[a, b]$. Use geometric reasoning to argue that the Trapezoidal Rule overestimates $\int_a^b f$. What if $f(x)$ is positive and $f''(x)$ is negative?

35. Suppose that $f(x)$ is a polynomial of degree no higher than 3. Why does Simpson's Rule give the exact value of $\int_a^b f$?

36. To complete the argument in the text for Simpson's Rule, we must prove that the area of the shaded region in Figure 2 is $\frac{1}{3}\Delta x(y_0 + 4y_1 + y_2)$. Do this as follows.
    (a) Let $p(x) = Ax^2 + Bx + C$ be an equation of the parabola passing through (x_0, y_0), (x_1, y_1), (x_2, y_2) and let $r = x_0$, $s = x_2$. Show that

    $$\int_r^s p(x)\,dx =$$

    $$\frac{1}{6}(s - r)[2A(s^2 + sr + r^2) + 3B(s + r) + 6C]$$

    (b) Noting that $x_1 = \frac{1}{2}(r + s)$, show that $y_0 + 4y_1 + y_2$ is the expression in brackets in part (a).
    (c) Combine parts (a) and (b) to finish the proof.

Additional Problems

Find each of the following integrals.

1. $\displaystyle\int \frac{x\,dx}{2 + x}$
2. $\displaystyle\int_3^5 \frac{dx}{x^2 - 6x + 13}$
3. $\displaystyle\int \cos\sqrt{x}\,dx$
4. $\displaystyle\int \frac{2 - \cos t}{\sin t}\,dt$
5. $\displaystyle\int \frac{2x^2 + 1}{x^3 + x}\,dx$
6. $\displaystyle\int_0^1 \tan^{-1}\sqrt{x}\,dx$
7. $\displaystyle\int_0^1 \ln(x^2 + 1)\,dx$
8. $\displaystyle\int \frac{e^x\,dx}{e^{2x} + 1}$
9. $\displaystyle\int \tan^5 x \sec x\,dx$
10. $\displaystyle\int \frac{x + 4}{x^2 + 4}\,dx$
11. $\displaystyle\int \frac{dx}{3\sin x + 4\cos x}$
12. $\displaystyle\int_0^{\ln 3} \frac{dx}{1 + e^x}$
13. $\displaystyle\int_0^{\pi/2} e^{-x}\cos x\,dx$
14. $\displaystyle\int_2^4 \frac{dx}{\sqrt{-x^2 + 6x - 5}}$
15. $\displaystyle\int_0^{\pi/2} \sin^5 t\cos^3 t\,dt$
16. $\displaystyle\int \frac{dx}{\cos x + \cot x}$
17. $\displaystyle\int_0^2 \frac{x^2\,dx}{\sqrt{16 - x^2}}$
18. $\displaystyle\int_1^4 \frac{dx}{x + 2\sqrt{x}}$
19. $\displaystyle\int x\sin^{-1} x^2\,dx$
20. $\displaystyle\int_0^{\ln 2} \frac{e^x\,dx}{9 - e^{2x}}$
21. $\displaystyle\int_0^5 \frac{x\,dx}{\sqrt{9 - x}}$
22. $\displaystyle\int \frac{dx}{\sqrt{x^2 - 4}}$
23. $\displaystyle\int x\sinh^{-1} x\,dx$
24. $\displaystyle\int_1^4 \frac{dx}{\sqrt{x}(1 + \sqrt{x})^4}$
25. $\displaystyle\int_0^1 x^3 e^{-x^2}\,dx$
26. $\displaystyle\int \frac{\sqrt{x^2 + 16}}{x^3}\,dx$
27. $\displaystyle\int \frac{x^2 - 1}{2x - 5}\,dx$
28. $\displaystyle\int_0^4 x^3\sqrt{16 - x^2}\,dx$
29. $\displaystyle\int_0^{\pi/4} \sec^2 x\tan^2 x\,dx$
30. $\displaystyle\int \frac{x^2 - 2}{x^3 - x^2}\,dx$
31. $\displaystyle\int_0^3 \frac{dx}{(x^2 + 9)^{3/2}}$
32. $\displaystyle\int \frac{x^2 + 1}{x^3 + 1}\,dx$
33. $\displaystyle\int_0^{\pi/2} \frac{dx}{2 + \sin x}$
34. $\displaystyle\int (x - \cos x)^2\,dx$
35. $\displaystyle\int \frac{dx}{x^2(x^2 + 1)}$
36. $\displaystyle\int_0^{\pi/6} \frac{dx}{\cos 2x}$
37. $\displaystyle\int \frac{x\,dx}{x^3 - 1}$
38. $\displaystyle\int_0^{\pi} e^x \sin x\,dx$

39. $\displaystyle\int_3^5 \frac{dx}{x^2 - 6x + 13}$

40. $\displaystyle\int_0^1 x^3 \cosh x \, dx$

41. $\displaystyle\int \frac{\tan^{-1} x}{x^2} \, dx$

42. $\displaystyle\int x\sqrt{x^4 - 16} \, dx$

43. $\displaystyle\int \frac{x \, dx}{(x-2)^3}$

44. $\displaystyle\int x^5 \sin x \, dx$

45. $\displaystyle\int \frac{dx}{5 + 4\cos x}$

46. Find
$$\int_0^4 \frac{x^3 \, dx}{\sqrt{x^2 + 9}}$$
by making an algebraic substitution.

47. Do Problem 46 by making a trigonometric substitution.

48. Use the identity $\sin^2(t/2) = \frac{1}{2}(1 - \cos t)$ to find
$$\int \frac{dt}{1 - \cos t}$$

49. Do Problem 48 by making the substitution $t = 2\tan^{-1} u$. (See Example 4, Section 10.5.)

50. Use the identity $\cos^2(t/2) = \frac{1}{2}(1 + \cos t)$ to find
$$\int \frac{dt}{1 + \cos t}$$

51. Do Problem 50 by making the substitution $t = 2\tan^{-1} u$.

Use the reduction formulas in Section 10.1 to evaluate each of the following integrals.

52. $\displaystyle\int_0^{\pi/2} \sin^5 x \, dx$

53. $\displaystyle\int_0^{\pi/2} \cos^7 x \, dx$

54. $\displaystyle\int_0^{\pi} \sin^6 \frac{x}{2} \, dx$

55. $\displaystyle\int_0^{\pi} \cos^8 \frac{x}{2} \, dx$

56. Show that
$$\int_0^1 \sinh^{-1} x \, dx = 1 - \sqrt{2} + \ln(1 + \sqrt{2})$$

57. Find the area of the region bounded by the curve $y = \tan^{-1} x$, the x axis, and the line $x = 1$.

58. The region enclosed by the curve $x = a\cos t$, $y = b\sin t$ is rotated about the x axis. Find the volume generated, as follows.
   (a) Use the parametric equations as they stand to show that
   $$V = 2\pi ab^2 \int_0^{\pi/2} \sin^3 t \, dt$$
   and evaluate the integral.
   (b) Eliminate the parameter before setting up an integral for V.
   (c) What is V if $a = b$?

59. In an AC circuit the *power* at time t is the product of current I and voltage V. Find the average power during one cycle in each of the following situations.
   (a) The current and voltage are in phase, that is,
   $$I = I_0 \cos(\omega t) \qquad \text{and} \qquad V = V_0 \cos(\omega t)$$
   (b) The current and voltage are out of phase, that is,
   $$I = I_0 \cos(\omega t) \qquad \text{and} \qquad V = V_0 \cos(\omega t + \alpha)$$
   where $0 < \alpha < \pi/2$.

In each of the following, use the given value of n in the Trapezoidal Rule to obtain an approximation to the integral.

60. $\displaystyle\int_0^{\pi} e^{\sin x} \, dx \qquad (n = 4)$

61. $\displaystyle\int_0^2 \frac{dx}{x^3 + 2} \qquad (n = 6)$

62. $\displaystyle\int_0^1 \frac{\tan x}{x} \, dx \qquad (n = 4)$

63. Do Problem 60 by using Simpson's Rule.

64. Do Problem 61 by using Simpson's Rule.

65. Do Problem 62 by using Simpson's Rule.