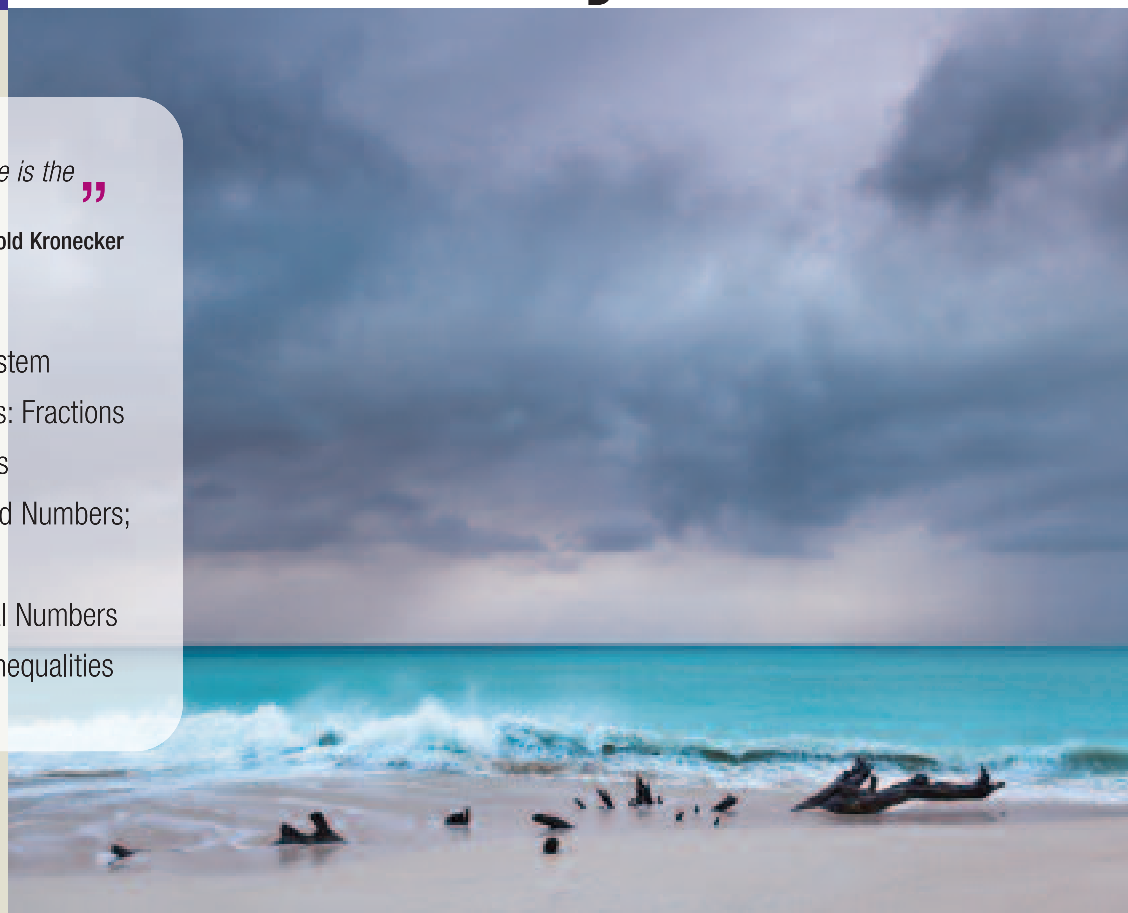
1 The Real Number System

"God made integers, all else is the work of man."

—Leopold Kronecker



Meteorologists are scientists who study weather phenomena, like hurricanes, hoping to be better able to predict their occurrences and effects. Like all scientists, they use numbers to measure, compare, and classify their information. For example, the Saffir-Simpson Hurricane Scale uses small ***natural numbers*** (1, 2, 3, 4, and 5) to classify hurricanes according to the amount of damage they are likely to cause. The recent Hurricane Katrina reached category 5 status. Scales like this often employ natural numbers because (as their name suggests) these are the types of numbers with which most people are comfortable. ***Inequalities*** may be used to express the range of wind speeds in miles per hour; for a category 5 storm, the speeds are .155 mph (greater than 155 mph). For a category 1 hurricane, the speeds are between 74 and 95 mph, which can be written as $74 < \text{wind speed} < 95$.

http://www.noaa.gov/

Scientists require other types of numbers too—for instance, **rational numbers** like 28.92 inches (the smallest central pressure for a category 1 hurricane). Many more examples for the uses of numbers and inequalities to describe hurricanes and other meteorological phenomena can be found at http://www.noaa.gov/.

What other uses do meteorologists and other specialists make of the real number system, number lines, inequalities, and operations with real numbers? The chapter project will suggest some ideas.

■ ■ ■

Arithmetic teaches us that the rule "two plus two equals four" is truly independent of the kind of objects to which the rule applies; it doesn't matter whether the objects are apples or ants, countries or cars. Observations such as this one led to the study of the properties of numbers in an abstract sense, that is, the study of those properties that apply to *all* numbers, regardless of what the numbers represent.

Since we deal in much of our work with the *real numbers*, our studies will begin with a review of the *real number system*. We then introduce symbols to denote arbitrary numbers, a practice characteristic of algebra. The remainder of the chapter is devoted to explaining some of the fundamental properties of the real number system.

1.1 The Real Number System

Although this text does not stress the set approach to algebra, the concept and notation of sets is useful at times.

Sets

A **set** is a collection of objects or numbers, which are called the **elements** or **members** of the set. The elements of a set are written within braces, so that

$$A = \{4, 5, 6\}$$

tells us that the set A consists of the numbers 4, 5, and 6. The set

$$B = \{\text{Squibb, Ford, Honeywell}\}$$

consists of the names of these three corporations. We also write $4 \in A$, which we read as "4 is a member of the set A" or "4 belongs to the set A." Similarly, Ford $\in B$ is read as "Ford is a member of the set B," and I.B.M. $\notin B$ is read as "I.B.M. is not a member of the set B."

If every element of a set A is also a member of a set B, then A is a **subset** of B. For example, the set of all robins is a subset of the set of all birds.

EXAMPLE 1 WORKING WITH SETS

The set C consists of the names of all coins whose denomination is less than 50 cents.

(a) Write C in set notation.
(b) Is dime $\in C$?
(c) Is half-dollar $\in C$?
(d) Is $H = \{\text{nickel, dime}\}$ a subset of C?

SOLUTIONS

(a) We have $C = \{\text{penny, nickel, dime, quarter}\}$

(b) yes (c) no (d) yes

✔ Progress Check 1

The set V consists of the vowels in the English alphabet.

(a) Write V in set notation.
(b) Is the letter k a member of V?
(c) Is the letter u a member of V?
(d) List the subsets of V having four elements.

Answers

(a) $V = \{a, e, i, o, u\}$ (b) no (c) yes
(d) $\{a, e, i, o\}, \{e, i, o, u\}, \{a, i, o, u\}, \{a, e, o, u\}, \{a, e, i, u\}$

The Real Number System

Much of our work in algebra deals with the set of real numbers. Let's review the composition of this number system.

The numbers 1, 2, 3, . . . , used for counting, form the set of **natural numbers**. If we had only these numbers to use to show the profit earned by a company, we would have no way to indicate that the company had no profit or had a loss. To indicate no profit we introduce 0, and for losses we need to introduce negative numbers. The numbers

$$\ldots, -2, -1, 0, 1, 2, \ldots$$

form the set of **integers**. Thus, every natural number is an integer. However, not every integer is a natural number.

When we try to divide two apples equally among four people, we find no number in the set of integers that will express how many apples each person should get. We need to introduce the **rational numbers**, which are numbers that can be written as a ratio of two integers,

$$\frac{p}{q} \qquad \text{with } q \text{ not equal to zero}$$

Examples of rational numbers are

$$0 \qquad \frac{2}{3} \qquad -4 \qquad \frac{7}{5} \qquad \frac{-3}{4}$$

Thus, when we divide two apples equally among four people, each person gets half, or $\frac{1}{2}$ an apple. Since every integer n can be written as $\frac{n}{1}$, we see that every integer is a rational number. The number 1.3 is also a rational number, since $1.3 = \frac{13}{10}$.

We have now seen three fundamental number systems: the natural number system, the system of integers, and the rational number system. Each system we have introduced includes the previous system or systems, and each is more complicated than the one before. However, the rational number system is still inadequate for sophisticated uses of mathematics since there exist numbers that are not rational, that is, numbers that cannot be written as the ratio of two integers. These are called **irrational numbers**. It can be shown that the number a that satisfies $a \cdot a = 2$ is such a number. The number π, which is the ratio of the circumference of a circle to its diameter, is also such a number.

The decimal form of a rational number will either terminate, as

$$\frac{3}{4} = 0.75 \qquad -\frac{4}{5} = -0.8$$

or will form a repeating pattern, as

$$\frac{2}{3} = 0.\underbrace{666}\ldots \qquad \frac{1}{11} = 0.\underbrace{090909}\ldots \qquad \frac{1}{7} = 0.\underbrace{1428571\ldots}.$$

Remarkably, the decimal form of an irrational number *never* forms a repeating pattern. Although we sometimes write $\pi = 3.14$, this is only an approximation, as is

$$\pi = 3.1415926536\ldots$$

Similarly, the decimal form of $\sqrt{2}$ can be approximated by $1.4142136\ldots$, which goes on forever and never forms a repeating pattern.

The rational and irrational numbers together form the **real number system** (Figure 1).

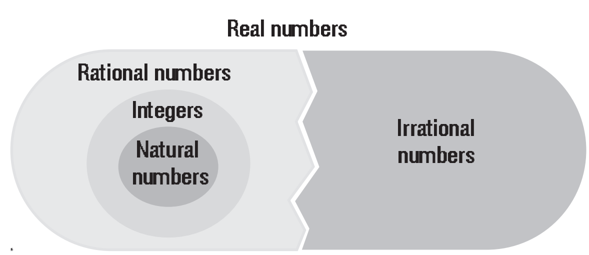


Figure 1 The Real Number System

The Real Number Line

To obtain a simple and useful geometric description of the set of real numbers we can draw a horizontal straight line, which we will call the **real number line**; pick a point, label it with the number 0, call it the **origin**, and denote it by O; and

Mnemonic Math

How I want a drink
Alcoholic of course
After the heavy lectures
Involving quantum mechanics!

This "poem" appeared in the *Science Times* on July 5, 1988. The same article contained yet another creative gem:

How I wish I could
Enumerate pi easily,
Since all these (censored) mnemonics
Prevent recalling any of pi's sequence
More simply

This second poem gives you a hint as to the "meaning" of these words. If you list the number of letters in each word, you have the sequence

3-1-4-1-5-9-2-6-5-3-5-8-9-7-9-3-2-3-8-4-5

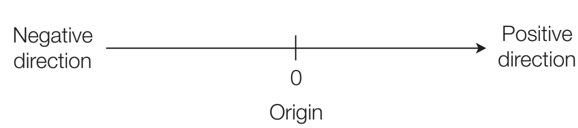
Now, stick a decimal point after the first digit and you have

3.14159265358979323845

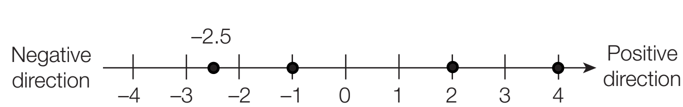
which should look vaguely familiar. This is the value of π accurate to 20 decimal places, and the "poems" are mnemonics recalling the value of this famous irrational number.

For centuries mathematicians have been fascinated by the challenge of calculating π. Gregory and David Chudnovsky of Columbia University recently became the first mathematicians to push the calculation of π to better than one billion places. And, should your interest run to poetry, π mnemonics exist in French, German, Greek, and Spanish.

then we choose the **positive direction** to the right of the origin and the **negative direction** to the left of the origin. An arrow indicates the positive direction.



Now we select a unit of length for measuring distance. With each positive real number r, we associate the point that is r units to the right of the origin; and with each negative number $-s$, we associate the point that is s units to the left of the origin. We can now show some points on the real number line.



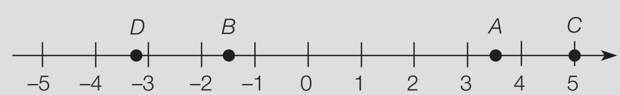
Conversely, let P be a point on the real number line. If P is to the right of the origin and r units from the origin, we associate the real number r with P. If P is to the left of the origin and s units from the origin, we associate the real number $-s$ with P.

Thus, the set of all real numbers is identified with the set of all points on a straight line. Every point on the line corresponds to a real number, called its **coordinate**; and for every real number there is a point on the line. We often say that the set of real numbers and the set of points on the real number line are in **one-to-one correspondence**. The numbers to the right of the origin are called **positive**. The numbers to the left of the origin are called **negative**. The positive numbers and zero together are called the **nonnegative** numbers, whereas the negative numbers and zero together are called the nonpositive numbers.

EXAMPLE 2 POINTS ON A REAL NUMBER LINE

Draw a real number line and plot the following points: $-\frac{3}{2}$, 2, $\frac{13}{4}$.

SOLUTION

> **✔ Progress Check 2**
>
> Determine the real numbers denoted on the real number line as A, B, C, and D.
>
> 
>
>
> **Answers**
>
> A: $\frac{7}{2}$ B: $-\frac{3}{2}$ C: 5 D: $-\frac{13}{4}$

Exercise Set 1.1

In Exercises 1–12 choose the correct answer(s) from the following: (a) rational number, (b) natural number, (c) real number, (d) integer, (e) irrational number.

1. The number 2 is
2. The number -3 is
3. The number $-\frac{2}{3}$ is
4. The number 0.8 is
5. The number 3 is
6. The numbers -1, -2, and -3 are
7. The numbers 0, 1, and 2 are
8. The numbers 0, $\frac{1}{2}$, 1, $\frac{2}{3}$, and $-\frac{4}{5}$ are
9. The numbers $\sqrt{2}$ and π are
10. The numbers 0.5 and 0.8 are
11. The numbers $\frac{\pi}{3}$ and 2π are
12. The numbers 0, $\frac{1}{2}$, $\sqrt{2}$, π, 4, and -4 are

In Exercises 13–21 determine whether the given statement is true (T) or false (F).

13. -14 is a natural number.

14. $-\frac{4}{5}$ is a rational number.

15. $\frac{\pi}{3}$ is a rational number.

16. 2.5 is an integer.

17. $\frac{1.75}{18.6}$ is an irrational number.

18. 0.75 is an irrational number.

19. $\frac{4}{5}$ is a real number.

20. 3 is a rational number.

21. $\sqrt{2}$ is a real number.

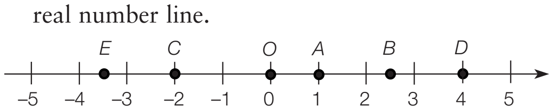
22. Draw a real number line and plot the following points.

    (a) 4 (b) 22 (c) $\dfrac{5}{2}$ (d) 23.5 (e) 0

23. Draw a real number line and plot the following points.

    (a) 25 (b) 4 (c) -3.5 (d) $\dfrac{7}{2}$ (e) 2π

24. Estimate the real number associated with the points A, B, C, D, O, and E on the accompanying real number line.



25. Represent each of the following by a positive or negative integer.
    (a) a profit of \$10
    (b) a loss of \$20
    (c) a temperature of 20° above zero
    (d) a temperature of 5° below zero

In Exercises 26–31 indicate which of the two given numbers appears first, viewed from left to right, on the real number line.

26. 4, 6
27. 2, 0
28. -2, 3
29. 0, -4
30. -5, -2
31. 4, -5

In Exercises 32–37 indicate which of the two given numbers appears second, viewed from left to right, on the real number line.

32. 9, 8
33. 0, 3
34. -4, 2
35. -4, 0
36. -3, -4
37. -2, 5

In Exercises 38–40 indicate the given set of numbers on the real number line.

38. The natural numbers less than 8

39. The natural numbers greater than 4 and less than 10

40. The integers that are greater than 2 and less than 7

41. Your graphing calculator can convert a decimal number to a fraction. However, when we input π, we get a decimal back:

Explain in your own words why this is the case.

42. A hurricane with a central pressure of 27.25 inches might be classified as a category 4 storm. What subset of the real numbers does this number belong to? Can you verify this using your graphing calculator (see Exercise 41)?

1.2 Arithmetic Operations: Fractions

The vocabulary of arithmetic carries over to our study of algebra. In multiplying two real numbers, each of the numbers is called a **factor** and the result is called the **product**.

$$\underbrace{6 \cdot 5}_{\text{factors}} = \underbrace{30}_{\text{product}}$$

Terminology and Order of Operations

In this text we will indicate multiplication by a dot, as in the example just given, or by parentheses.

$$(6)(5) = 30$$

We will avoid use of the multiplication sign $\times$, since it may be confused with other algebraic symbols.

Let's look at the terminology of division.

$$\underbrace{30}_{\text{dividend}} \div \underbrace{6}_{\text{divisor}} = \underbrace{5}_{\text{quotient}}$$

Most of the time we write division this way:

$$\frac{30}{6} = 5$$

Numbers to be multiplied have the same name (factors), whereas in division the numbers have different names (dividend, divisor). This suggests that we can interchange the factors in multiplication without altering the product, but that interchanging the dividend and divisor in division will alter the quotient. In general:

> Addition can be performed in any order.
> Multiplication can be performed in any order.
> Subtraction must be performed in the given order.
> Division must be performed in the given order.

What happens when more than one operation appears in a problem? To avoid ambiguity, we adopt the following simple rule:

> Always do multiplication and division before addition and subtraction.

EXAMPLE 1 ORDER OF ARITHMETIC OPERATIONS

Perform the indicated operations.

(a) $2 + (3)(4) - 5 = 2 + 12 - 5 = 9$

(b) $\dfrac{10}{5} + 7 + 3 \cdot 6 = 2 + 7 + 18 = 27$

✔ Progress Check 1

Perform the indicated operations.

(a) $2 \cdot 4 - 6 + 3$ (b) $(3)(4)\ 2\ 2\ 1\ \dfrac{9}{3}$

Answers

(a) 5 (b) 13

We will provide complete rules for the order of operations in the next section.

Fractions

It is important to master the *arithmetic* of fractions since this serves as background for the *algebra* of fractions. We prefer to write $10 \div 5$ in the form $\frac{10}{5}$, which we call a **fraction**. The number above the line is called the **numerator;** that below the line is called the **denominator**.

$$\begin{array}{ll}\text{numerator} & 10 \\ \hline \text{denominator} & 5\end{array}$$

Multiplication of fractions is straightforward.

Multiplication of Fractions

Step 1. Multiply the numerators of the given fractions to find the numerator of the product.

Step 2. Multiply the denominators of the given fractions to find the denominator of the product.

EXAMPLE 2 MULTIPLICATION OF FRACTIONS

Multiply.

(a) $\dfrac{3}{5} \cdot \dfrac{7}{2}$ (b) $\dfrac{2}{9} \cdot \dfrac{5}{3} \cdot 4$

SOLUTIONS

(a) $\dfrac{3}{5} \cdot \dfrac{7}{2} = \dfrac{3 \cdot 7}{5 \cdot 2} = \dfrac{21}{10}$

(b) $\dfrac{2}{9} \cdot \dfrac{5}{3} \cdot 4 = \dfrac{2}{9} \cdot \dfrac{5}{3} \cdot \dfrac{4}{1} = \dfrac{2 \cdot 5 \cdot 4}{9 \cdot 3 \cdot 1} = \dfrac{40}{27}$

✔ Progress Check 2

Multiply.

(a) $\frac{4}{3} \cdot \frac{7}{3}$ (b) $\frac{5}{12} \cdot \frac{7}{3} \cdot \frac{1}{2}$

Answers

(a) $\frac{28}{9}$ (b) $\frac{35}{72}$

Reciprocal

The **reciprocal** of $\frac{3}{4}$ is found by inverting $\frac{3}{4}$ to obtain $\frac{4}{3}$. Thus, $\frac{4}{3}$ is the reciprocal of $\frac{3}{4}$. Similarly, $\frac{1}{5}$ is the reciprocal of 5, and $\frac{1}{\pi}$ is the reciprocal of π. Note that the product of a real number and its reciprocal is always equal to 1. The number 0 does not have a reciprocal since the product of 0 and any number is 0.

- Every real number, except 0, has a reciprocal such that the product of the number and its reciprocal is equal to 1.
- The number 0 does not have a reciprocal, and *division by 0 is not permitted.*

Division of fractions can always be converted into a multiplication problem by forming the reciprocal.

Division of Fractions

Step 1. Invert the divisor.

Step 2. Multiply the resulting fractions.

Since 0 has no reciprocal, division by 0 is not defined.

EXAMPLE 3 DIVISION OF FRACTIONS

Divide.

(a) $\frac{4}{9} \div \frac{3}{5}$ (b) $\dfrac{\frac{2}{3}}{\frac{5}{7}}$

SOLUTIONS

(a) $\frac{4}{9} \div \frac{3}{5} = \frac{4}{9} \cdot \frac{5}{3} = \frac{4 \cdot 5}{9 \cdot 3} = \frac{20}{27}$ (b) $\dfrac{\frac{2}{3}}{\frac{5}{7}} = \frac{2}{3} \div \frac{5}{7} = \frac{2}{3} \cdot \frac{7}{5} = \frac{14}{15}$

✔ Progress Check 3

Divide.

(a) $\frac{8}{7}$ 4 $\frac{3}{2}$ (b) $\frac{\frac{1}{2}}{3}$ (c) $\frac{2}{11}$ 4 $\frac{5}{3}$

Answers

(a) $\frac{16}{21}$ (b) $\frac{1}{6}$ (c) $\frac{6}{55}$

The same fractional value can be written in many ways. Thus,

$$\frac{3}{2} = \frac{6}{4} = \frac{18}{12} = \frac{72}{48}$$

are four forms of the same fractional value.

Equivalent Fractions

The value of a fraction is not changed by multiplying or dividing both the numerator and denominator by the same number (other than 0). The result is called an **equivalent fraction**.

If we multiply a fraction, say $\frac{5}{2}$ by $\frac{3}{3}$, we are really multiplying by a "disguised" equivalent form of 1, since $\frac{3}{3} = 1$. Thus,

$$\frac{5}{2} = \frac{5}{2} \cdot \frac{3}{3} = \frac{15}{6}$$

EXAMPLE 4 FINDING AN EQUIVALENT FRACTION

Find the equivalent fraction.

$$\frac{7}{3} = \frac{?}{12}$$

SOLUTION

Since $12 = 3 \cdot 4$, we multiply the original denominator by 4 to obtain the new denominator. Then we must also multiply the numerator by 4.

$$\frac{7}{3} \cdot \frac{4}{4} = \frac{7 \cdot 4}{3 \cdot 4} = \frac{28}{12}$$

✔ Progress Check 4

Find the equivalent fraction.

(a) $\frac{5}{4} = \frac{?}{20}$ (b) 6 5 $\frac{?}{4}$ (c) $\frac{2}{3} = \frac{8}{?}$

Answers

(a) $\frac{25}{20}$ (b) $\frac{24}{4}$ (c) $\frac{8}{12}$

Finding all the Primes up to *N*: The Sieve of Eratosthenes

Prime Integers Less Than or Equal to 100

2	3	~~4~~	5	~~6~~	7	~~8~~
~~9~~	~~10~~	11	~~12~~	13	~~14~~	~~15~~
~~16~~	17	~~18~~	19	~~20~~	~~21~~	~~22~~
23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29
~~30~~	31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~
37	~~38~~	~~39~~	~~40~~	41	~~42~~	43
~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	57
~~58~~	59	~~60~~	61	~~62~~	~~63~~	~~64~~
~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~	71
~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~
79	~~80~~	~~81~~	~~82~~	83	~~84~~	~~85~~
~~86~~	87	~~88~~	89	~~90~~	~~91~~	~~92~~
~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~
~~100~~						

An integer p . 1 is called a prime if the only positive integers that divide p are p and 1. For example, 3, 5, 11, and 2 are primes. The number 2 is the only even prime, since every even integer greater than 2 is divisible by 2. A positive integer that is not a prime is said to be a composite. For example, 4, 10, and 15 are composite integers.

A method for listing all the primes up to a given integer N was developed by the Greek scientist and mathematician Eratosthenes (275–194 B.C.), who was a friend of Archimedes. We will describe this method, called the Sieve of Eratosthenes, and apply it to the accompanying table, which lists the positive integers less than or equal to 100.

Step 1. Make a list of all integers from 2 to N.

Step 2. Since 2 is the first prime, cross out all multiples of 2. The next integer in the list that has not been crossed out is 3, which is a prime. Now cross out all multiples of 3. The next integer in the list that has not been crossed out is 5, which is a prime. Next, cross out all multiples of 5. Repeat the process until the list is exhausted.

Step 3. The numbers that have not been crossed out are the primes less than N.

You probably noticed that no additional cross-outs occurred after you crossed out the multiples of 7. In general, you can stop when you reach a number K such that K times K is at least N.

The number of computations required for executing the "sieve" rises dramatically as N increases. For this reason, the "sieve" has become a favorite benchmark program for comparing computer hardware and software.

Now let's reverse the process. We saw that $\frac{7}{3}$ can be written as

$$\frac{7}{3} \cdot \frac{4}{4} = \frac{28}{12}$$

which is equivalent to $\frac{7}{3}$. Beginning with $\frac{28}{12}$, we can write the numerator and denominator as a product of factors:

$$\frac{28}{12} = \frac{7 \cdot 4}{3 \cdot 4} = \frac{7}{3} \cdot \frac{4}{4} = \frac{7}{3} \cdot 1 = \frac{7}{3}$$

We say that $\frac{7}{3}$ is the **reduced form** of $\frac{28}{12}$. This illustrates the **cancellation principle.**

Cancellation Principle

Common factors appearing in both the numerator and denominator of a fraction can be canceled without changing the value of the fraction. When a fraction has no common factors in its numerator and denominator, it is said to be in reduced form.

EXAMPLE 5 OBTAINING REDUCED FORM

Write $\frac{15}{27}$ in reduced form.

SOLUTION

$$\frac{15}{27} = \frac{5 \cdot 3}{9 \cdot 3} = \frac{5}{9} \cdot \frac{3}{3} = \frac{5}{9} \cdot 1 = \frac{5}{9}$$

✔ Progress Check 5

Write in reduced form.

(a) $\frac{22}{60}$ (b) $\frac{90}{15}$ (c) $\frac{32}{12}$

Answers

(a) $\frac{11}{30}$ (b) 6 (c) $\frac{8}{3}$

WARNING

Only multiplicative factors common to both the *entire* numerator and the *entire* denominator can be canceled. *Don't* write

$$\frac{6+5}{3} = \frac{\overset{2}{\cancel{6}}+5}{\cancel{3}} = \frac{7}{1} = 7$$

Since 3 is not a multiplicative factor common to the *entire* numerator, we may not cancel.

The addition and subtraction of fractions can sometimes be more complicated than their multiplication. We begin by stating the key idea for adding and subtracting fractions.

Addition and Subtraction Principle

We can add or subtract fractions directly only if they have the same denominator.

When fractions do have the same denominator, the process is easy: Add or subtract the numerators and keep the common denominator. Thus,

$$\frac{3}{4} + \frac{15}{4} = \frac{3+15}{4} = \frac{18}{4} = \frac{9}{2}$$

Least Common Denominator

If the fractions we wish to add or subtract do not have the same denominator, we must rewrite them as equivalent fractions that do have the same denominator. There are easy ways to find the **least common denominator** (**LCD**) of two or more fractions, that is, the smallest number that is divisible by each of the given denominators.

To find the LCD of two or more fractions, say, $\frac{1}{2}$, $\frac{5}{6}$, and $\frac{4}{9}$, we first write each denominator as a product of prime numbers. Recall that a **prime number** is a natural number greater than 1 whose only factors are itself and 1. For example, 5 is a prime number, since it is divisible only by 5 and 1. Other examples of numbers written as a product of primes are:

$$2 = 2$$
$$6 = 2 \cdot 3$$
$$9 = 3 \cdot 3$$

We then form a product in which each distinct prime factor appears the greatest number of times that it occurs in any single denominator. This product is the LCD. In our example, the prime factor 2 appears, at most, once in any denominator, while the prime factor 3 appears twice in a denominator. Thus, the LCD is $2 \cdot 3 \cdot 3 = 18$.

The LCD is the tool we need to add or subtract fractions with different denominators. Here is an example of the process.

EXAMPLE 6 USING THE LCD IN THE ADDITION OF FRACTIONS

Find the following sum:

$$\frac{2}{5} + \frac{3}{4} - \frac{2}{3}$$

SOLUTION

Addition and Subtraction of Fractions

Step 1. Find the LCD of the fractions.

$$\text{LCD} = 5 \cdot 2 \cdot 2 \cdot 3 = 60$$

Step 2. Convert each fraction to an equivalent fraction with the LCD as its denominator.

$$\frac{2}{5} = \frac{2}{5} \cdot \frac{12}{12} = \frac{24}{60}$$
$$\frac{3}{4} = \frac{3}{4} \cdot \frac{15}{15} = \frac{45}{60}$$
$$\frac{2}{3} = \frac{2}{3} \cdot \frac{20}{20} = \frac{40}{60}$$

Step 3. The fractions now have the same denominator. Add and subtract the numerators as indicated.

$$\frac{2}{5} + \frac{3}{4} - \frac{2}{3} = \frac{24}{60} + \frac{45}{60} - \frac{40}{60}$$
$$= \frac{24 + 45 - 40}{60}$$
$$= \frac{29}{60}$$

Step 4. Write the answer in reduced form.

Step 4.

Answer: $\frac{29}{60}$

✔ Progress Check 6

Find the sum $\frac{2}{3} + \frac{4}{7}$.

Answer

$\frac{26}{21}$

EXAMPLE 7 ADDING AND SUBTRACTING FRACTIONS

Perform the indicated operations and simplify.

(a) $\frac{1}{2} - \frac{1}{6} + \frac{1}{3}$ (b) $\dfrac{\frac{4}{3} - \frac{1}{1}}{\frac{1}{3} + \frac{3}{4}}$

SOLUTIONS

(a) We see that the LCD is 6 and

$$\frac{1}{2} = \frac{1}{2} \cdot \frac{3}{3} = \frac{3}{6}$$

$$\frac{1}{6} = \frac{1}{6} \cdot \frac{1}{1} = \frac{1}{6}$$

$$\frac{1}{3} = \frac{1}{3} \cdot \frac{2}{2} = \frac{2}{6}$$

Thus,

$$\frac{1}{2} - \frac{1}{6} + \frac{1}{3} = \frac{3}{6} - \frac{1}{6} + \frac{2}{6} = \frac{3 - 1 + 2}{6} = \frac{4}{6} = \frac{2}{3}$$

(b) The LCD of the fractions in the numerator is 6, and hence the numerator is

$$\frac{4}{3} - \frac{1}{2} = \frac{4}{3} \cdot \frac{2}{2} - \frac{1}{2} \cdot \frac{3}{3} = \frac{8}{6} - \frac{3}{6} = \frac{5}{6}$$

The LCD of the fractions in the denominator is 12, and hence the denominator is

$$\frac{1}{3} + \frac{3}{4} = \frac{1}{3} \cdot \frac{4}{4} + \frac{3}{4} \cdot \frac{3}{3} = \frac{4}{12} + \frac{9}{12} = \frac{13}{12}$$

Then the given fraction is

$$\dfrac{\frac{5}{6}}{\frac{13}{12}} = \frac{5}{\cancel{6}} \cdot \frac{\overset{2}{\cancel{12}}}{13} = \frac{10}{13}$$

✔ Progress Check 7

Perform the indicated operations and simplify.

(a) $\dfrac{3}{2} + \dfrac{5}{9} - \dfrac{1}{3}$ (b) $\dfrac{\frac{2}{3}+\frac{5}{6}}{\frac{1}{2}+\frac{2}{3}}$

Answers

(a) $\dfrac{31}{18}$ (b) $\dfrac{9}{7}$

Percent

Percent is a way of writing a fraction whose denominator is 100. The percent sign, %, following a number means "place the number over 100." Thus, 7% means $\frac{7}{100}$.

A fraction whose denominator is 100 is converted to decimal form by moving the decimal point in the numerator two places to the left and eliminating the denominator.

$$\frac{65}{100} = 0.65$$

Since a percent is understood to mean a fraction whose denominator is 100, we see that

$$7\% = 0.07$$

Similarly, we change a decimal to a percent by moving the decimal point two places to the right and adding the percent sign.

$$0.065 = 6.5\%$$

To write a fraction as a percent, we multiply the fraction by 100 and divide the new fraction by its denominator. For example, to write $\frac{1}{20}$ as a percent, we form

$$\frac{100}{20} = 5$$

so $\frac{1}{20} = 5\%$. Similarly, to write $\frac{1}{7}$ as a percent, we form

$$\frac{100}{7} = 14.28$$

so $\frac{1}{7} = 14.28\%$.

EXAMPLE 8 CONVERTING PERCENTS, DECIMALS, AND FRACTIONS

Write each percent as a decimal and as a fraction, and each decimal or fraction as a percent.

(a) 25% (b) 142% (c) 0.06 (d) 2.1 (e) $\frac{3}{4}$ (f) $\frac{21}{5}$ (g) $\frac{1}{6}$

SOLUTIONS

(a) $25\% = 0.25;\ 25\% = \frac{25}{100} = \frac{1}{4}$ (b) $142\% = 1.42;\ 142\% = \frac{142}{100} = \frac{71}{50}$

(c) $0.06 = 6\%$ (d) $2.1 = 210\%$

(e) $\frac{3}{4} = \frac{75}{100} = 75\%$ (f) $\frac{21}{5} = \frac{420}{100} = 420\%$

(g) $\frac{1}{6} = 0.167$ (rounded) 5 16.7%

✔ Progress Check 8

Write each percent as a decimal and as a fraction, and each decimal or fraction as a percent.

(a) 62.5% (b) $\frac{1}{2}\%$ (c) 0.26 (d) 3.475

(e) $\frac{1}{8}$ (f) $\frac{5}{2}$ (g) $\frac{1}{9}$

Answers

(a) 0.625, $\frac{5}{8}$ (b) 0.005, $\frac{1}{200}$ (c) 26% (d) 347.5%

(e) 12.5% (f) 250% (g) 11.1%

It is common practice to state business problems in terms of percent. You have heard and read statements such as:

> Ms. Smith was promised an 8% salary increase.
> The Best Savers Bank pays 5.75% interest per year.
> Automobile prices will increase 4.62% on July 1.
> During the sale period, all merchandise is reduced by 20%.

To find the **percent of a number**, we must convert the percent to a decimal or fraction and then *multiply* by the number.

EXAMPLE 9 FINDING THE PERCENT OF A NUMBER

(a) What is 30% of 15?

(b) What is 5% of 400?

(c) A bank pays 6.75% interest per year. What will the annual interest be on a deposit of \$500?

(d) The price of a refrigerator selling at \$600 is to be reduced by 20%. What is the sale price?

SOLUTIONS

(a) $30\% = 0.3$ and $(0.3)(15) = 4.5$

(b) $5\% = 0.05$ and $(0.05)(400) = 20$

(c) $6.75\% = 0.0675$ and $(0.0675)(\$500) = \$33.75$

(d) $20\% = 0.2$ and $(0.2)(\$600) = \$120 =$ amount of discount

$$\begin{aligned}\text{Sale price} &= \text{original price} - \text{discount}\\ &= \$600 - \$120\\ &= \$480\end{aligned}$$

✔ Progress Check 9

(a) What is 40% of 60?

(b) What is 2% of 1200?

(c) How much interest will be earned during one year on a deposit of \$2500 at 7.5% per year?

(d) The price of an automobile selling at \$6800 will be increased by 4%. What is the new price?

Answers

(a) 24 (b) 24 (c) \$187.50 (d) \$7072

Exercise Set 1.2

In Exercises 1–14 perform the indicated operations.

1. $\dfrac{2(6+2)}{4}$

2. $\dfrac{(4+5)6}{18}$

3. $\dfrac{6(3+1)}{2} + 3 \cdot 5$

4. $\dfrac{8(6-1)}{4} - 3 \cdot 2$

5. $\dfrac{(4+5)(2+3)}{3} - 3 \cdot 5$

6. $\dfrac{(7-2)(8-2)}{3} + 7 \cdot 4$

7. $\dfrac{2}{11} \cdot \dfrac{10}{3} \cdot \dfrac{2}{5}$

8. $\dfrac{7}{5} \cdot \dfrac{4}{3} \cdot 2$

9. $\dfrac{\frac{2}{3}}{\frac{1}{5}}$

10. $\dfrac{\frac{3}{4}}{\frac{4}{3}}$

11. $\dfrac{\frac{2}{5}}{\frac{3}{10}}$

12. $\dfrac{\frac{1}{2}t}{\frac{5}{6}}$

13. $\frac{2}{3} \div \frac{4}{9}$

14. $\frac{3}{5} \div \frac{9}{25}$

In Exercises 15–20 find the number that makes the fractions equivalent.

15. $\frac{4}{3} = \frac{?}{9}$

16. $\frac{3}{4} = \frac{15}{?}$

17. $1 = \frac{7}{?}$

18. $2 = \frac{?}{14}$

19. $\frac{5}{4} = \frac{?}{20}$

20. $\frac{4}{7} = \frac{12}{?}$

In Exercises 21–24 find the least common denominator of the given fractions.

21. $\frac{1}{4}, \frac{1}{2}, \frac{2}{15}$

22. $\frac{1}{3}, \frac{1}{9}, \frac{1}{5}$

23. $\frac{1}{20}, \frac{1}{30}, \frac{1}{45}$

24. $2, \frac{1}{4}, \frac{5}{36}$

In Exercises 25–32 perform the indicated operations and simplify.

25. $\frac{3}{4} + \frac{2}{3}$

26. $\frac{2}{3} + \frac{5}{6}$

27. $\frac{1}{4} + \frac{2}{3} - \frac{1}{2}$

28. $\frac{1}{5} - \frac{1}{2} + \frac{1}{3}$

29. $\dfrac{\frac{1}{6}+\frac{1}{2}}{\frac{5}{4}+\frac{2}{3}}$

30. $\dfrac{\frac{1}{2}-\frac{3}{8}}{\frac{1}{3}+\frac{1}{4}}$

31. $\dfrac{2-\frac{1}{3}}{3+\frac{1}{4}}$

32. $\dfrac{\frac{3}{5}-\frac{1}{10}}{1+\frac{1}{2}}$

In Exercises 33–42 change the given percent to both fractional and decimal forms.

33. 20%

34. 60%

35. 65.5%

36. 32.5%

37. 4.8%

38. 5.5%

39. 120%

40. 160%

41. $\frac{1}{5}\%$

42. $\frac{1}{10}\%$

In Exercises 43–54 convert each number to a percent.

43. 0.05

44. 0.03

45. 0.425

46. 0.345

47. 6.28

48. 7.341

49. $\frac{3}{5}$

50. $\frac{5}{6}$

51. $\frac{9}{4}$

52. $\frac{6}{5}$

53. $\frac{2}{7}$

54. $\frac{4}{3}$

55. What is 35% of 60?

56. What is 60% of 80?

57. What is 140% of 30?

58. What is 160% of 50?

59. What is $\frac{1}{2}\%$ of 40?

60. What is $\frac{2}{5}\%$ of 20?

In Exercises 61–64 use a calculator to convert each number to decimal form and then determine the smallest number.

61. $\frac{5}{32}, \frac{11}{64}, \frac{8}{50}$

62. $\frac{20}{27}, \frac{28}{39}, \frac{78}{103}$

63. $\frac{150}{171}, \frac{78}{88}, \frac{125}{144}$

64. $\frac{814}{653}, \frac{910}{731}, \frac{3875}{3107}$

In Exercises 65–68 use a calculator to convert each number to a percent and determine the largest number.

65. $\frac{32}{55}, \frac{15}{27}, \frac{52}{89}$

66. $\frac{75}{61}, \frac{941}{765}, \frac{66}{54}$

67. $\frac{137}{49}, \frac{850}{304}, \frac{267}{96}$

68. $\frac{999}{1077}, \frac{67}{73}, \frac{297}{321}$

69. A bank pays 7.25% interest per year. If a depositor has \$800 in a savings account, how much interest will the bank pay him at the end of one year?

70. Suppose that you buy a \$5000 General Motors bond that pays 9.3% interest per year. What will be the amount of the dividend when mailed to you by G.M. at the end of the year?

71. A savings bank pays 8% interest per year. If a depositor has \$6000 in a savings account, what will be the amount in the account at the end of one year, assuming that no withdrawals are made?

72. A student has borrowed \$3000 at a rate of 7% per year. How much interest is owed to the bank at the end of one year?

73. A record store embarks on an advertising campaign to raise its profits by 20%. If this year's profits were \$96,000, what will next year's profits be if the campaign succeeds?

*74. An \$800 stereo system will be sold at a 25% reduction. What will be the new price?

*75. In order to cope with rising costs, an oil producer plans to raise prices by 15%. If a barrel of oil now sells for \$28.00, what will be the new price?

*76. A boat that originally sold for \$600 is now on sale for \$540. What is the percentage of discount?

*77. The holder of an \$8000 savings certificate gets a \$640 check at the end of the year. What is the annual rate of simple interest?

*78. A department store runs the following sale on a brand of stereo equipment. During the first week of the sale, the merchandise is discounted by 10%. If the merchandise is not sold by the second week, it is discounted by 20% of the sale price that was in effect during the first week. (This is called **chain discount.**) What is the price during the second week of a receiver that originally sold for \$400?

*79. On September 1, an automobile manufacturer introduces the new model of a car and increases by 6% the price that was in effect for the car on August 31. Since demand for the new model exceeds supply, on October 1 the manufacturer raises the price by 2% of the price in effect on September 30. What is the new price of a car that on August 31 sold for \$8000? (This is known as **chain percent increase.**)

80. Use your graphing calculator to convert each number to a fraction in lowest terms (see your manual for instructions): 0.12, 0.1278, $0.\overline{63}$ (this is 0.636363 . . .) how many repetitions must you enter before your calculator will interpret this as a repeating decimal?

81. According to the Bipartisan Task Force on Funding Disaster Relief (1995), hurricanes accounted for 39.9% of insurance payouts for the period 1984–1993. For every million dollars paid out in this period, how many dollars were paid out for hurricanes?

82. Agricultural damages accounted for 1.52 billion dollars out of a total of 30 billion dollars of damages related to hurricane Andrew, which struck Florida in August 1992. What percentage of the total damages were agricultural?

Source for data in Exercises 81–82: Pielke Jr., Roger A., and Roger A. Pielke Sr. *Hurricanes: Their Nature and Impacts on Society.* Chichester: John Wiley & Sons, 1997.

1.3 Algebraic Expressions

A rational number is one that can be written as p/q where p and q are integers (and q is not zero). These symbols can take on more than one distinct value. For example, when $p = 5$ and $q = 7$, we have the rational number $\frac{5}{7}$; when $p = -3$ and $q = 2$, we have the rational number $\frac{-3}{2}$. The symbols p and q are called **variables**, since various values can be assigned to them.

If we invest P dollars at an interest rate of 6% per year, we will have $P + 0.06P$ dollars at the end of the year. We call $P + 0.06P$ an **algebraic expression.** Note that an algebraic expression involves **variables** (in our case, P), **constants** (such as 0.06), and **algebraic operations** (such as $+, -, \times, \div$). Virtually everything we do in algebra involves algebraic expressions, sometimes as simple as our example and sometimes very complicated.

When we assign a value to each variable in an algebraic expression and carry out the indicated operations, we are "evaluating" the expression.

EXAMPLE 1 EVALUATING AN ALGEBRAIC EXPRESSION

Evaluate.

(a) $2x + 5$ when $x = 3$ (b) $\dfrac{3m + 4n}{m + n}$ when $m = 3, n = 2$

SOLUTIONS

(a) Substituting 3 for x, we have

$$2(3) + 5 = 6 + 5 = 11$$

(b) Substituting, we have

$$\frac{3(3) + 4(2)}{3 + 2} = \frac{9 + 8}{3 + 2} = \frac{17}{5}$$

✔ Progress Check 1

Evaluate.

(a) $\dfrac{r + 2s}{s - r}$ when $r = 1, s = 3$

(b) $4x + 2y - z$ when $x = 1, y = 4, z = 2$

Answers

(a) $\dfrac{7}{2}$ (b) 10

Order of Operations

If we want to evaluate

$$3 + \frac{5y}{2} \quad \text{when } y = 4$$

in what order should we perform the operations? The interpretation

$$\frac{3 + 5y}{2}$$

doesn't produce the same result as the interpretation

$$3 + \frac{5y}{2}$$

Fortunately, most calculators and computer languages use the same convention for determining the order of operations. This order or **hierarchy of operations** is shown below.

Hierarchy of Operations

1. Perform all operations within parentheses.
2. Compute any squares or powers (see Section 1.4).
3. Perform multiplication and division, working from left to right.
4. Perform addition and subtraction, working from left to right.

To evaluate an expression, substitute the given values for the variables and then apply the rules for the hierarchy of operations.

EXAMPLE 2 EVALUATING AN ALGEBRAIC EXPRESSION

Evaluate the expression

$$\frac{3(x-1)}{y+2} + (x-y)(x+y) \quad \text{when } x = 5,\ y = 2$$

SOLUTION

Substituting, we have

$$\begin{aligned} \frac{3(5-1)}{(2+2)} + (5-2)(5+2) &= \frac{3(4)}{4} + (3)(7) && \text{Operations in parentheses} \\ &= 3 + 21 && \text{Multiplication and division} \\ &= 24 && \text{Addition} \end{aligned}$$

✔ Progress Check 2

Evaluate.

(a) $\dfrac{2(x-1)}{(x+1)+(x+3)}$ when $x = 2$

(b) $1 + (2 - x) + \dfrac{y}{y-1}$ when $x = 1,\ y = 2$

Answers

(a) $\dfrac{1}{4}$ (b) 4

Exercise Set 1.3

In Exercises 1–4 determine whether the given statement is true (T) or false (F).

1. $3x + 2 = 8$ when $x = 2$
2. $5x - 1 = 11$ when $x = 2$
3. $2xy = 12$ when $x = 2,\ y = 3$
4. $\dfrac{2x+y}{7} = 7$ when $x = 3,\ y = 1$

In Exercises 5–12 evaluate the given expression when $x = 4$.

5. $2x + 3$
6. $3x - 2$
7. $\frac{1}{2}x$
8. $3(x - 1)$
9. $(2x)(2x)$
10. $(2x + 1)x$
11. $\dfrac{1}{2x+3}$
12. $\dfrac{x}{2x-4}$

In Exercises 13–20 evaluate the given expression when $a = 3,\ b = 4$.

13. $2a + b$
14. $3a - b$
15. $2b - a$
16. ab
17. $3(a + 2b)$
18. $\dfrac{1}{3a+b}$
19. $\dfrac{2a-b}{b}$
20. $\dfrac{a+b}{b-a}$

In Exercises 21–28 evaluate the given expression when $r = 2,\ s = 3,\ t = 4$.

21. $r + 2s + t$
22. rst

23. $\dfrac{rst}{r + s + t}$

24. $(r + s)t$

2t5. $\dfrac{r + s}{t}$

26. $\dfrac{r + s + t}{t}$

27. $\dfrac{t - r}{rs}$

28. $\dfrac{3(r + s + t)}{s}$

29. Evaluate $2\pi r$ when $r = 3$. (Recall that π is approximately 3.14.)

30. Evaluate $\frac{9}{5}C + 32$ when $C = 37$.

31. Evaluate $0.02r + 0.314st + 2.25t$ when $r = 2.5$, $s = 3.4$, and $t = 2.81$.

32. Evaluate $10.421x + 0.821y + 2.34xyz$ when $x =$ 3.21,
$y = 2.42$, and $z = 1.23$.

In Exercises 33–36 evaluate the given expression when $x = -3.25$ and $y = 0.75$.

33. $3 + x \div 6 \cdot x - 1$

34.$3 \div (x + y) - y$

35. $2 \cdot y + 1 \div (x + 1)$

36. $-1 + (2x - y) \div 2 + y \div (1 - x)$

*37. If P dollars are invested for t years at a simple interest rate of r percent per year, the amount on hand at the end of t years is $P + Prt$. Suppose \$2000 is invested at 8% per year ($r = 0.08$). How much money is on hand after

(a) one year (b) three years
(c) half a year (d) eight months

(*Hint:* Express eight months as a fraction of a year.).

*38. The perimeter of a rectangle is given by the formula
$P = 2(L + W)$, where L is the length and W is the width of the rectangle. Find the perimeter if

(a) $L = 2$ feet, $W = 3$ feet.
(b) $L = \frac{1}{2}$ meter, $W = \frac{1}{4}$ meter.
(c) $L = 13$ inches, $W = 15$ inches.

39. Al purchases 3 boxes of cereal at \$1.25 per box, 2 bottles of soda at \$1.10 per bottle, and $1\frac{1}{2}$ pounds of salad at \$2.29 per pound. Write an expression for the total cost.

40. If apples cost a cents per pound, pears cost p cents per pound, and cucumbers cost c cents each, write an expression for the total bill when Joan purchases 2 pounds of apples, $3\frac{1}{2}$ pounds of pears, and 6 cucumbers.

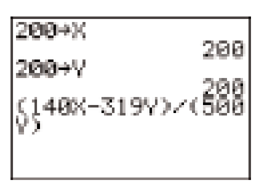
41. The campus cafeteria sells hamburgers for \$1.35, hot dogs for 85¢, and french fries for 80¢ per order. Write an expression for the total cost when Jim purchases d hot dogs, h hamburgers, and f orders of french fries.

42. Hi-Fi house is offering a 10% discount on all record and cassette purchases. Write an expression for the total cost when a student purchases N cassettes at \$5.99 each, M cassettes at \$6.99 each, and K records at \$6 each.

43. Use your calculator to evaluate the expression

$$\frac{140x - 319y}{500y}$$

for $x = 200$, $y = 200$. First, store the values under variable names. Then, enter the expression using those variable names. Now you can store new values and retrieve the expression you entered (it should not be necessary to enter it again). Evaluate the same expression for $x = 210$ and $y = 199$.



1.4 Operating with Signed Numbers; Exponents

Let's review the rules for operating with signed numbers before using them in more complicated problems.

We say that $-a$ is the **opposite** of a if a and $-a$ are equidistant from the origin and lie on opposite sides of the origin.

Since $-a$ is opposite in direction to a, then $-(-a)$ must have the same direction as a. We see that

$$-(-a) = a$$

We also refer to $-a$ as the **negative** of a. Of course, the negative of a number need not be negative: $-(-5) = 5$.

Here are the rules for addition with signed numbers:

Addition with Like Signs

Ignoring the signs, add the numbers. The sign of the answer is the same as the common sign of the original numbers.

$$5 + 2 = 7$$
$$(-5) + (-2) = -7$$

Addition with Unlike Signs

Ignoring the signs, find the difference of the numbers. The sign of the answer is the sign of the number that is larger in absolute value.

$$6 + (-4) = 2 \qquad 3 + (-5) = -2$$
$$(-7) + 3 = -4 \qquad (-5) + 8 = 3$$

EXAMPLE 1 ADDING SIGNED NUMBERS

(a) $2 + 4 = 6$ (b) $2 + (-4) = -2$ (c) $(-2) + (-4) = -6$

(d) $(-2) + 4 = 2$ (e) $3 + [(-5) + (-2)] = 3 + (-7) = -4$

(f) $[(-4) + (-7)] + 3 = -11 + 3 = -8$ (g) $8 + 0 = 8$

✔ Progress Check 1

Add.

(a) $(-3) + (-7)$ (b) $(-5) + 1$ (c) $2 + (-6)$

(d) $-2 + [5 + (-4)]$ (e) $0 + (-8)$ (f) $[3 + (-6)] + (-1)$

Answers

(a) -10 (b) -4 (c) -4

(d) -1 (e) -8 (f) -4

Subtraction problems can be converted into addition of signed numbers.

Subtraction

Change $a - b$ to $a + (-b)$ and follow the rules for addition of signed numbers.

$$
\begin{aligned}
5 - 2 &= 5 + (-2) = 3 \\
-3 - 4 &= -3 + (-4) = -7 \\
2 - (-6) &= 2 + (+6) = 8 \\
-6 - (-5) &= -6 + (+5) = -1
\end{aligned}
$$

EXAMPLE 2 SUBTRACTING SIGNED NUMBERS

(a) $7 - 4 = 7 + (-4) = 3$ (b) $10 - (-6) = 10 + (+6) = 16$

(c) $-8 - 2 = -8 + (-2) = -10$ (d) $-5 - (-4) = -5 + (+4) = -1$

(e) $-7 - (-7) = -7 + (+7) = 0$ (f) $(-2 - 5) + 6 = [-2 + (-5)] + 6 = 21$

✔ Progress Check 2

Perform the operations.

(a) $3 - 8$ (b) $-6 - 7$ (c) $-9 - (-5)$ (d) $16 - (-9)$

(e) $(14 - 5) - 4$ (f) $(-11 + 2) - 4$ (g) $(-6 - 4) - 2$

Answers

(a) -5 (b) -13 (c) -4 (d) 25

(e) 5 (f) -13 (g) -12

The rules for determining the sign in multiplication and division are straightforward.

Multiplication and Division

If both numbers have the same sign, the result is positive. If the numbers have opposite signs, the result is negative.

$$3 \cdot 4 = 12 \qquad \frac{6}{3} = 2$$

$$(-2)(-5) = 10 \qquad \frac{-8}{-4} = 2$$

$$(-4)(6) = -24 \qquad \frac{-10}{2} = -5$$

$$(7)(-3) = -21 \qquad \frac{12}{-3} = -4$$

EXAMPLE 3 MULTIPLYING AND DIVIDING SIGNED NUMBERS

(a) $4 \cdot \left(-\frac{1}{5}\right) = -\frac{4}{5}$ (b) $\left(-\frac{2}{3}\right)(-3) = 2$ (c) $\frac{-4}{8} = -\frac{1}{2}$

(d) $\frac{-16}{-24} = \frac{2}{3}$ (e) $(-5) \cdot \frac{1}{4} = -\frac{5}{4}$ (f) $\frac{18}{-2} = -9$

✔ Progress Check 3

(a) $(23)\left(\frac{2}{-7}\right)$ (b) $\left(\frac{2}{3}\right)\left(\frac{-3}{4}\right)$ (c) $\frac{20}{-6-4}$

(d) $\frac{4-5}{3-6}$ (e) $(-4)\left(\frac{2-3}{4}\right)$ (f) $\left(\frac{1}{5}\right)\left(\frac{5-6}{4}\right)$

Answers

(a) $\frac{6}{7}$ (b) $-\frac{1}{2}$ (c) -2

(d) $\frac{1}{3}$ (e) 1 (f) $-\frac{1}{20}$

Let's apply the rules for operating with signed numbers to the evaluation of algebraic expressions.

EXAMPLE 4 EVALUATING AN ALGEBRAIC EXPRESSION

Evaluate the given expression when $x = -1$, $y = -1$.

(a) $2x + \frac{x-1}{y+2}$ (b) $-3(2x - y) + (-4)(2y - x)$

SOLUTIONS

(a) Substituting, we have

$$2(-1) + \frac{(-1 - 1)}{-1 + 2} = -2 + \frac{-2}{1} = -2 + (-2) = -4$$

(b) Substituting, we have

$$\begin{aligned} &-3[2(-1) - (-1)] + (-4)[2(-1) - (-1)] \\ &\quad = -3(-2 + 1) + (-4)(-2 + 1) \\ &\quad = -3(-1) + (-4)(-1) = 3 + 4 = 7 \end{aligned}$$

✔ Progress Check 4

Evaluate the given expression when $x = 2$, $y = -1$.

(a) $-(-y)$ (b) $2 - 3x + y$ (c) $\dfrac{2 - 2x}{2 - 2y}$ (d) $\dfrac{x + y}{x - y}$

Answers

(a) -1 (b) -5 (c) $-\dfrac{1}{2}$ (d) $\dfrac{1}{3}$

Mathematicians use a special notation to indicate a product of repeated factors. For example, the product

$$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$$

is written as

$$3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 3^5$$

We call 3 the **base** and 5 the **exponent**. We can generalize this notation by the following definition.

Exponent Notation

For any real number a and natural number n,

$$a^n = a \cdot a \cdot a \cdots a$$

EXAMPLE 5 WORKING WITH EXPONENT NOTATION

Evaluate.

(a) 2^6 (b) 0.1^3 (c) $(-2)^4$ (d) $(-5)^3$

SOLUTIONS

(a) $2^6 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 64$ (b) $0.1^3 = (0.1)(0.1)(0.1) = 0.001$

(c) $(-2)^4 = (-2)(-2)(-2)(-2) = 16$ (d) $(-5)^3 = (-5)(-5)(-5) = -125$

✔ Progress Check 5

Evaluate.

(a) 10^3 (b) $(-4)^4$ (c) $(-2)^5$ (d) 3^4 (e) $\left(\frac{2}{3}\right)^3$

Answers

(a) 1000 (b) 256 (c) -32 (d) 81 (e) $\frac{8}{27}$

Exercise Set 1.4

In Exercises 1–54 simplify the given expression by carrying out the indicated operations.

1. $3 + 5$
2. $-2 + (-3)$
3. $(-3) + (-4)$
4. $2 + (-3)$
5. $4 + (-2)$
6. $-4 + 6$
7. $-4 + 2$
8. $0 + (-2)$
9. $3 + 0$
10. $3 + (-3)$
11. $5 - 3$
12. $5 - 8$
13. $5 - (-3)$
14. $4 - (-4)$
15. $-8 - (-3)$
16. $-6 - (-7)$
17. $5 - (-6)(-3)$
18. $4 - (-1)(2)$
19. $[-3 - (-2)] - 1$
20. $[-8 - (-4)] - 5$
21. $(-4 - 2) - (-3)$
22. $2\left(\frac{3}{4}\right)$
23. $(-2)(-5)$
24. $(-3)\left(-\frac{8}{6}\right)$
25. $\left(-\frac{5}{6}\right)\left(\frac{9}{15}\right)$
26. $\left(\frac{3}{5}\right)\left(-\frac{10}{4}\right)$
27. $\frac{8}{2}$
28. $\frac{-10}{-2}$
29. $\frac{-15}{5}$
30. $\frac{20}{-4}$
31. $\frac{-15}{25}$
32. $\frac{15}{-\frac{3}{4}}$
33. $\frac{-12}{-\frac{2}{3}}$
34. $(-4)\left(-\frac{5}{2}\right)$
35. $\frac{3}{5}\left(-\frac{15}{2}\right)$
36. $\left(-\frac{3}{4}\right)0$
37. $\left(-\frac{4}{5}\right)\left(-\frac{15}{2}\right)$
38. $-(-2)$
39. $\frac{4 - 4}{2}$
40. $\frac{14 + 1}{-5 - (-2)}$
41. $\frac{5 + (-5)}{3}$
42. $\frac{-18}{-3 - 6}$
43. $\frac{15}{2 - 7}$
44. $\frac{24}{2 - 8}$
45. $\frac{-8 - 4}{3}$
46. $-5(2 - 4)$
47. $-4(4 - 1)$
48. $\frac{3(-5 + 1)}{-4(2 - 6)}$
49. $-(-2x + 3y)$
50. $(-x)(-y)$
51. $\frac{-x}{-y}$
52. $\frac{-x}{\frac{1}{2}}$
53. $\frac{2}{\frac{x}{-2}}$
54. $\frac{-a}{(-b)(-c)}$

In Exercises 55–57 evaluate the given expression when $x = -2$.

55. $x - 5$
56. $-2x$
57. $\frac{x}{x - 1}$

In Exercises 58–60 evaluate the given expression when $x = -3$, $y = -2$.

58. $x + 2y$
59. $x - 2y$
60. $\frac{4x - y}{y}$

In Exercises 61–68 evaluate a^n for the given values of a and n.

61. $a = -1, n = 5$

62. $a = 4, n = 2$

63. $a = 6, n = 3$

64. $a = -2, n = 4$

65. $a = \frac{2}{5}, n = 3$

66. $a = 0.2, n = -3$

67. $a = \frac{5}{3}, n = 4$

68. $a = -1, n = 100$

69. Subtract 3 from -5.

70. Subtract -3 from -4.

71. Subtract -5 from -2.

72. Subtract -2 from 8.

73. At 2 P.M. the temperature is 10°C above zero, and at 11 P.M. it is 2°C below zero. How many degrees has the temperature dropped?

74. Repeat the previous exercise if the temperature at 2 P.M. is 8°C and it is -4°C at 11 P.M.

75. A stationery store had a loss of \$400 during its first year of operation and a loss of \$800 during its second year. How much money did the store lose during the first two years of its existence?

76. A bicycle repair shop had a profit of \$150 for the month of July and a loss of \$200 for the month of August. How much money did the shop gain or lose over the two-month period?

*77. E. & E. Fabrics had a loss of x dollars during its first business year and a profit of y dollars its second year. Write an expression for the net profit or loss after two years.

*78. S. & S. Hardware had a profit of x dollars, followed by a loss that exceeded twice the profit by \$200. Write an expression for the net loss.

*79. The Student Stereo Shoppe had a loss of \$200 during its first year of business, a profit of \$800 during its second year, and a profit of \$900 during its third year. What was the average profit (or loss) over the three-year period?

80. Use your calculator to evaluate the expression

$$3200x - 2900y$$

for $x = -2900$, $y = 3200$. First, store the values under variable names. Then, enter the expression using those variable names. Now you can store new values and retrieve the expression you entered (it should not be necessary to enter it again). Evaluate the same expression for $x = -3200$ and $y = -2900$.

1.5 Properties of Real Numbers

The real numbers obey laws that enable us to manipulate algebraic expressions with ease. We'll use the letters a, b, and c to denote real numbers.

To begin, note that the sum of two real numbers is a real number and the product of two real numbers is a real number. These are known as the **closure properties**.

Closure Properties

Property 1. The sum of a and b, denoted by $a + b$, is a real number.

Property 2. The product of a and b, denoted by $a \cdot b$ or ab, is a real number.

We say that the set of real numbers is **closed** with respect to the operations of addition and multiplication, since the sum and product of two real numbers are also real numbers.

We know that

$$3 + 4 = 7 \qquad\qquad 3 \cdot 4 = 12$$
$$\text{and}$$
$$4 + 3 = 7 \qquad\qquad 4 \cdot 3 = 12$$

That is, we may *add or multiply real numbers in any order.* Writing this in algebraic symbols, we have the following:

Commutative Properties

Property 3. $a + b = b + a$	Commutative property of addition
Property 4. $ab = ba$	Commutative property of multiplication

EXAMPLE 1 USING THE COMMUTATIVE PROPERTIES

(a) $5 + 7 = 7 + 5$; $5 \cdot 7 = 7 \cdot 5$

(b) $3 + (-6) = -6 + 3$; $3 \cdot (-6) = (-6) \cdot 3$

(c) $3x + 4y = 4y + 3x$; $(3x)(4y) = (4y)(3x)$

✔ Progress Check 1

Use the commutative properties to write each expression in another form.

(a) $(-3) + 6$ (b) $(-4) \cdot 5$ (c) $-2x + 6y$ (d) $\left(\frac{3}{2}x\right)\left(\frac{1}{2}y\right)$

Answers

(a) $6 + (-3)$ (b) $5 \cdot (-4)$ (c) $6y + (-2x)$ (d) $\left(\frac{1}{2}y\right)\left(\frac{3}{2}x\right)$

When we add $2 + 3 + 4$, does it matter in what order we group the numbers? No. We see that

$$(2 + 3) + 4 = 5 + 4 = 9$$

and

$$2 + (3 + 4) = 2 + 7 = 9$$

Similarly, for multiplication of $2 \cdot 3 \cdot 4$ we have

$$(2 \cdot 3) \cdot 4 = 6 \cdot 4 = 24$$

and

$$2 \cdot (3 \cdot 4) = 2 \cdot 12 = 24$$

Clearly, *when adding or multiplying real numbers, we may group them in any order.* Translating into algebraic symbols, we have the following:

Associative Properties

Property 5. $(a + b) + c = a + (b + c)$	Associative property of addition
Property 6. $(ab)c = a(bc)$	Associative property of multiplication

EXAMPLE 2 USING THE ASSOCIATIVE PROPERTIES

(a) $5 + (2 + 3) = (5 + 2) + 3 = 10$ (b) $5 \cdot (2 \cdot 3) = (5 \cdot 2) \cdot 3 = 30$

(c) $(3x + 2y) + 4z = 3x + (2y + 4z)$ (d) $3(4y) = (3 \cdot 4)y = 12y$

(e) $(-2)[(-5)(-x)] = [(-2)(-5)](-x) = 10(-x) = -10x$

✔ Progress Check 2

Use the associative properties to simplify.

(a) $3 + (2 + x)$ (b) $6 \cdot 2xy$

Answers

(a) $5 + x$ (b) $12xy$

We can combine the commutative and associative properties to simplify algebraic expressions.

EXAMPLE 3 SIMPLIFYING AN EXPRESSION

Use the commutative and associative properties to simplify.

(a) $(3 + x) + 5$ (b) $\left(\frac{2}{3}y\right)\left(\frac{3}{4}\right)$ (c) $(2x - 4) + 7$

SOLUTIONS

(a) $(3 + x) + 5 = (x + 3) + 5$ Commutative property of addition

$= x + (3 + 5)$ Associative property of addition

$= x + 8$

(b) $\left(\frac{2}{3}y\right)\left(\frac{3}{4}\right) = \frac{3}{4} \cdot \left(\frac{2}{3}y\right)$ Commutative property of multiplication

$= \left(\frac{3}{4} \cdot \frac{2}{3}\right)y$ Associative property of multiplication

$= \frac{1}{2}y$

(c) $(2x - 4) + 7 = [2x + (-4)] + 7$

$= 2x + (-4) + 7$ Associative property of addition

$= 2x + 3$

✔ Progress Check 3

Use the commutative and associative properties to simplify.

(a) $4 + (2x + 2)$ (b) $\left(\frac{4}{5}x\right)\left(\frac{10}{2}\right)$ (c) $(5 - 3x) + 6$

Answers

(a) $6 + 2x$ (b) $4x$ (c) $11 - 3x$

The **distributive properties** deal with both addition and multiplication. For instance,

$$2(3 + 4) = 2(7) = 14 \qquad \text{and} \qquad (1 + 2)5 = (3)5 = 15$$

We may notice that

$$2(3) + 2(4) = 6 + 8 = 14$$

and

$$1(5) + 2(5) = 5 + 10 = 15$$

produce the same results as the first two equations. The distributive properties tell us that this is not an accident; rather, it is a rule that we may always use.

Distributive Properties

Property 7. $a(b + c) = ab + bc$

Property 8. $(a + b)c = ac + bc$

The distributive properties can be extended to factors that are a sum of more than two terms. Thus,

$$\begin{aligned} 3(5x + 2y - 4z) &= 3(5x) + 3(2y) + 3(-4z) \\ &= 15x + 6y - 12z \end{aligned}$$

EXAMPLE 4 USING THE DISTRIBUTIVE PROPERTIES

(a) $4(2x + 3) = 4(2x) + 4(3) = 8x + 12$

(b) $(4x + 2)6 = (4x)(6) + (2)(6) = 24x + 12$

(c) $\begin{aligned}[t] 2(x + 5y - 2z) &= 2x + 2(5y) + 2(-2z) \\ &= 2x + 10y - 4z \end{aligned}$

(d) $-(2y - x) = (-1)(2y - x) = (-1)(2y) + (-1)(2x) = -2y + x$

Note that a negative sign in front of parentheses is treated as multiplication by -1.

✔ Progress Check 4

Simplify, using the distributive properties.

(a) $5(3x + 4)$ (b) $(x + 3)7$ (c) $-2(3a - b + c)$

Answers

(a) $15x + 20$ (b) $7x + 21$ (c) $-6a + 2b - 2c$

It is easy to show that the commutative and associative properties *do not* hold for subtraction and division. For example,

$$2 - 5 = -3 \quad \text{but} \quad 5 - 2 = 3$$

and, in general,

$$a - b = -(b - a) \neq b - a$$

Similarly, $12 \div 3 \neq 3 \div 12$ shows that the commutative property does not hold for division.

The student is encouraged to provide counterexamples to show that the associative property does not hold for subtraction and division (see Exercise 19 in Exercise Set 1.5.)

Table 1 summarizes some of the important properties of real numbers.

TABLE 1 PROPERTIES OF REAL NUMBERS

For all real numbers a, b, and c:

Closure properties	$a + b$ is a real number. ab is a real number.
Commutative properties	$a + b = b + a$ $ab = ba$
Associative properties	$(a + b) + c = a + (b + c)$ $(ab)c = a(bc)$
Distributive properties	$a(b + c) = ab + ac$ $(a + b)c = ac + bc$
Identity properties	There is a unique real number 0 called the **additive identity** such that $a + 0 = a \quad \text{and} \quad 0 + a = a$ There is a unique real number 1 called the **multiplicative identify** such that $a \cdot 1 = a \quad \text{and} \quad 1 \cdot a = a$
Inverse properties	For every real number a there is a unique number $-a$ called the **additive inverse** or **opposite** of a such that $a + (-a) = 0$ For every real number a (other than 0) there is a unique real number $\frac{1}{a}$ called the **multiplicative inverse** or **reciprocal** such that $a \cdot \frac{1}{a} = 1$

Exercise Set 1.5

In Exercises 1–18 justify the given equation by using one or more properties of real numbers.

1. $2 + 5 = 5 + 2$
2. $(3 \cdot 2)(-4) = 3[(2)(-4)]$
3. $-2 \cdot 5 = 5(-2)$
4. $2(4 + 5) = 2 \cdot 4 + 2 \cdot 5$
5. $(4 + 3)2 = 4 \cdot 2 + 3 \cdot 2$
6. $3(2 - 4) = 3 \cdot 2 - 3 \cdot 4$
7. $-2(4 - 5) = (-2)(4) - 2(-5)$
8. $(3 + 2) + 4 = 3 + (2 + 4)$
9. $-3 + (2 + 5) = (-3 + 2) + 5$
10. $(2 - 5) + 8 = 2 + (-5 + 8)$
11. $3 + a = a + 3$
12. $2(x + 2) = 2x + 4$
13. $2(ab) = (2a)b$
14. $5(a + b) = 5(b + a)$
15. $2(xy) = x(2y)$
16. $4(a + b) = 4b + 4a$
17. $5 + (a + 2) = 2 + (5 + a)$
18. $(5x)y = 5(yx)$
19. Give examples showing that the commutative and associative properties do not hold for the operation of subtraction.

In Exercises 20–25 find and correct the mistake.

20. $a + 2a = 2a^2$
21. $2(a + 2) = 2a + 2$
22. $3(x - 2) = x - 6$
23. $(a - b)2 = 2a - b$
24. $3(ab) = (3a)(3b)$
25. $(2a + 3) + a = 3(a + 2)$

In Exercises 26–54 simplify the given expression.

26. $(2 + x) + 4$
27. $(2 - x) + 2$
28. $(x - 3) - 4$
29. $(x - 5) - 2x$
30. $(2x)(-5)$
31. $(-3x)(-4)$
32. $2(-3x)$
33. $a(2b)(3c)$
34. $4\left(\frac{3}{2}a\right)$
35. $\frac{2}{3}(9 + 12a - 6b)$
36. $\frac{4x}{-2}$
37. $\frac{-2}{4x}$
38. $\frac{-8x}{-4}$
39. $(3x)\left(-\frac{4}{9}y\right)$
40. $\frac{1}{4}(4a)$
41. $\frac{1}{5}(10ab)$
42. $\frac{4(b + 2)}{3}$
43. $\frac{3(5x - y)}{12}$
44. $6x + \frac{(y - 1)4}{2}$
45. $3a + \frac{(b - 3c)}{2}(-5)$
46. $3(a - 2) - 2(b + 4)$
47. $4(x + y) - 2(z - 2w)$
48. $3\left(\frac{2u + v}{6}\right) + \frac{1}{2}(4w)$
49. $2\left(\frac{x}{2} + y - 2\right) - (-2u + 4)$
50. $4\left(\frac{a - 2b + 4}{2}\right) + \frac{1}{3}(6c + 9)$
51. $-3.65\left(\frac{0.47 - 2.79}{6.44}\right)$
52. $\frac{6.92}{4.7}\left(\frac{2.01}{1.64 - 3.53}\right)$
53. $0.40\left(\frac{17.52 - 6.48 + 2.97}{3.60}\right) - 0.25(-4.75 + 2.92)$
54. $16.33\left(\frac{14.94}{3.87} - \frac{2.22 + 7.46}{2.96}\right)$

In Exercises 55–58 find a counterexample for each given statement; that is, find real values for the variables that make the statement false.

*55. $a(b + c) = ab + c$

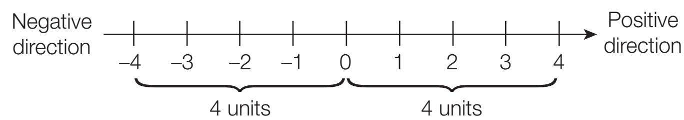
*56. $\frac{a}{b} = \frac{b}{a}$

*57. $(b - c)a = b - ca$

*58. $(a + b)(c + d) = ac + bd$

1.6 Absolute Value and Inequalities

When we introduced the real number line, we pointed out that positive numbers lie to the right of the origin and negative numbers lie to the left of the origin.



Suppose we are interested in the *distance* between the origin and the points labeled 4 and -4. Each of these points is four units from the origin, that is, the *distance is independent of the direction.*

When we are interested in the size of a number a and don't care about the direction or sign, we use the notation of **absolute value**, which we write as $|a|$. Thus,

$$|4| = 4$$
$$|-4| = 4$$

EXAMPLE 1 WORKING WITH ABSOLUTE VALUE

(a) $|-6| = 6$ (b) $|17.4| = 17.4$ (c) $|0| = 0$

(d) $|4 - 9| = |-5| = 5$ (e) $\left|\frac{2}{5} - \frac{6}{5}\right| = \left|-\frac{4}{5}\right| = \frac{4}{5}$

✔ Progress Check 1

Find the values.

(a) $|22|$ (b) $\left|-\frac{2}{7}\right|$ (c) $|4 - 4|$ (d) $|6 - 8|$ (e) $\left|\frac{1}{7} - \frac{3}{7}\right|$

Answers

(a) 22 (b) $\frac{2}{7}$ (c) 0 (d) 2 (e) $\frac{2}{7}$

The absolute value bars act as grouping symbols. We must work inside these grouping symbols before we can remove them.

EXAMPLE 2 ORDER OF OPERATIONS WITH ABSOLUTE VALUE

(a) $|-3| + |-6| = 3 + 6 = 9$

(b) $|3 - 5| - |8 - 6| = |-2| - |2| = 2 - 2 = 0$

(c) $\dfrac{|4-7|}{|-6|} = \dfrac{|-3|}{|-6|} = \dfrac{3}{6} = \dfrac{1}{2}$

(d) $\left|\dfrac{2-8}{3}\right| = \left|\dfrac{-6}{3}\right| = |-2| = 2$

✔ Progress Check 2

Find the values.

(a) $|-2| - |-4|$ (b) $\dfrac{|2-5|}{-3}$ (c) $\left|\dfrac{1-5}{2-8}\right|$

(d) $\dfrac{|-3| - |-6|}{4 - |-10|}$

Answers

(a) -2 (b) -1 (c) $\dfrac{2}{3}$ (d) $\dfrac{1}{2}$

The absolute value of a number is, then, always nonnegative. What can we do with the absolute value of a variable, say, $|x|$? We don't know whether x is positive or negative, so we can't write $|x| = x$. For instance, when $x = -4$, we have

$$|x| = |-4| = 4 \neq x$$

and when $x = 4$, we have

$$|x| = |4| = 4 = x$$

We must define absolute value so that it works for both positive and negative values of a variable.

Absolute Value

$$|a| = \begin{cases} a & \text{when } a \text{ is 0 or positive} \\ -a & \text{when } a \text{ is negative} \end{cases}$$

When a is positive, say, $a = 4$, the absolute value is the number itself; when a is negative, say, $a = -4$, the absolute value is the negative of a, or $+4$. Thus, the absolute value is always nonnegative.

The following properties of absolute value follow from the definition.

Properties of Absolute Value

For all real numbers a and b,

1. $|a|$ is nonnegative
2. $|a| = |-a|$
3. $|a - b| = |b - a|$

We began by showing how absolute value can be used to denote distance from the origin without regard to direction. We will conclude by demonstrating the use of absolute value to denote the distance between any two points a and b on the real number line. In Figure 2, the distance between the points labeled 2 and 5 is 3 units and can be obtained by evaluating either $|5 - 2|$ or $|2 - 5|$. Similarly, the distance between the points labeled -1 and 4 is given by either $|4 - (-1)| = 5$ or $|-1 - 4| = 5$. Using the notation to denote the distance between the points A and B, we provide the following definition.

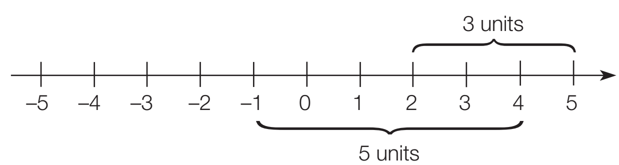


Figure 2 Distance Expressed by Absolute Value

Distance on the Real Number Line

> The distance $\overline{AB}$ between points A and B on the real number line, whose coordinates are a and b, respectively, is given by
>
> $$\overline{AB} = |b - a|$$

Property 3 then tells us that $\overline{AB} = |b - a| = |a - b|$. Viewed another way, Property 3 states that the distance between any two points on the real number line is independent of the direction.

EXAMPLE 3 ABSOLUTE VALUE AND DISTANCE

Let points A, B, and C have coordinates -4, -1, and 3, respectively, on the real number line, and let the origin, with coordinate 0, be denoted by O. Find the following distances.

(a) $\overline{AB}$ (b) $\overline{CB}$ (c) $\overline{OB}$

SOLUTIONS

Using the definition, we have

(a) $\overline{AB} = |-1 - (-4)| = |-1 + 4| = |3| = 3$

(b) $\overline{CB} = |-1 - 3| = |-4| = 4$

(c) $\overline{OB} = |-1 - 0| = |-1| = 1$

✔ Progress Check 3

The points P, Q, and R on the real number line have coordinates -6, 4, and 6, respectively. Find the following distances.

(a) $\overline{PR}$ (b) $\overline{QP}$ (c) $\overline{PQ}$ (d) $\overline{PO}$

Answers

(a) 12 (b) 10 (c) 10 (d) 6

Inequalities

If a and b are real numbers, we can compare their positions on the real number line by using the relations of **less than, greater than, less than or equal to**, and **greater than or equal to**, denoted by the inequality symbols $<$, $>$, $\leqslant$, and $\geqslant$, respectively. Table 2 describes both algebraic and geometric interpretations of the inequality symbols.

Here is a helpful way to remember the meaning of the symbols $>$ and $<$. We can think of the symbols $>$ and $<$ as pointers that always point to the lesser of the two numbers.

EXAMPLE 4 USING THE INEQUALITY SYMBOLS

(a) $2 < 5$ (b) $-1 < 3$ (c) $6 > 4$

✔ Progress Check 4

Make a true statement by replacing the square with the symbol $<$ or $>$.

(a) $7 \square 10$ (b) $16 \square 8$ (c) $4 \square 22$

Answers

(a) $<$ (b) $>$ (c) $>$

TABLE 2 INEQUALITIES AND THE REAL NUMBER LINE

Algebraic Statement	Equivalent Statement	Location on the Number Line
$a > b$	a is greater than b. or b is less than a.	a lies to the right of b.
$a > 0$	a is greater than zero. or a is positive.	a lies to the right of the origin.
$a < b$	a is less than b. or b is greater than a.	a lies to the left of b.
$a < 0$	a is less than zero. or a is negative.	a lies to the left of the origin.
$a \geqslant b$	a is greater than or equal to b. or b is less than or equal to a.	a coincides with b or lies to the right of b.
$a \geqslant 0$	a is greater than or equal to zero. or a is nonnegative.	a coincides with or lies to the right of the origin.
$a \leqslant b$	a is greater than or equal to b. or b is greater than or equal to a.	a coincides with b or lies to the left of b.
$a \leqslant 0$	a is less than or equal to zero. or a is nonpositive.	a coincides with or lies to the left of the origin.

We can use the real number line to illustrate the relations $<$ and $>$. For example, in Figure 3 we show that the inequality $x < 3$ is satisfied by *all* points to the left of 3.

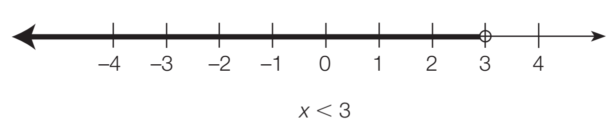


Figure 3 Graph of $x < 3$

Similarly, in Figure 4 we show that the inequality $x \geqslant -1$ is satisfied by *all* points to the right of (and including) -1.

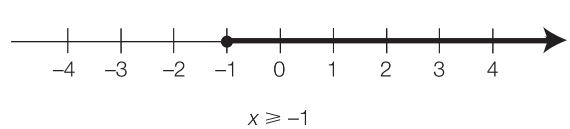
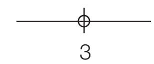
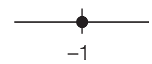


Figure 4 Graph of $x \geqslant -1$

For $x < 3$, the point labeled 3 does not satisfy the inequality; we indicate this by an open circle.



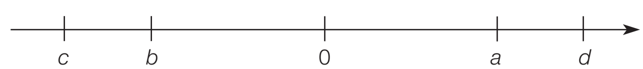
For $x \geqslant -1$, the point labeled -1 does satisfy the inequality; we indicate this by a solid circle.



In Figures 3 and 4, the shading indicates the "set" of all points whose coordinates satisfy the given inequality. The set of all such coordinates is called the **solution set** of the inequality, and we are said to have **graphed** the solution set of the inequality or to have graphed the inequality.

EXAMPLE 5 INEQUALITIES AND THE REAL NUMBER LINE

In the following figure,



(a) $a > b$, since a is to the right of b.
(b) $c < a$, since c is to the left of a.
(c) $b < 0$, since b is to the left of 0.
(d) $d > a$, since d is to the right of a.

✔ Progress Check 5

For the figure in Example 5, make a true statement by replacing each square with the symbol $<$ or $>$.

(a) $b \square d$ (b) $a \square c$ (c) $d \square 0$ (d) $b \square a$

Answers

(a) $<$ (b) $>$ (c) $>$ (d) $<$

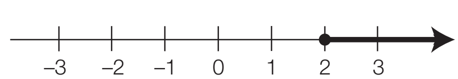
EXAMPLE 6 GRAPHING AN INEQUALITY

Graph the solution set of the inequality on the real number line.

(a) $x \geqslant 2$ (b) $x < 2$

SOLUTIONS

(a)



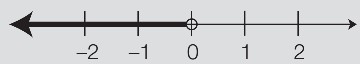
(b)

✔ Progress Check 6

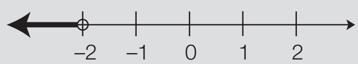
Graph the inequality on the real number line.

(a) $x < 0$ (b) $x \geqslant -1$ (c) $x < -2$

Answers

(a) 

(b) 

(c) 

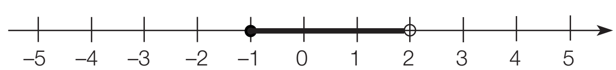
We also write compound inequalities, such as

$$-1 \leqslant x < 2$$

The solution set to this inequality consists of all real numbers that satisfy

$$-1 \leqslant x \quad \text{and} \quad x < 2$$

that is, all numbers between -1 and 2 and including -1 itself. We can easily graph the solution set on a real number line.

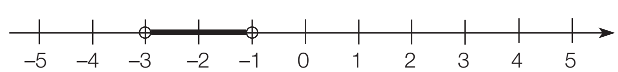


$-1 \leqslant x < 2$

EXAMPLE 7 GRAPHING A COMPOUND INEQUALITY

Graph $-3 < x < -1$, x a real number.

SOLUTION



✔ Progress Check 7

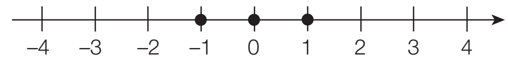
Graph $-2 \leqslant x \leqslant 3$, x a real number.

Answer

We can also graph the solution set to the inequality

$$-1 \leqslant x < 2, \ x \text{ an integer}$$

that is, the set of integers greater than or equal to -1 and less than 2. The solution set is $\{-1, 0, 1\}$.



EXAMPLE 8 GRAPHING A COMPOUND INEQUALITY

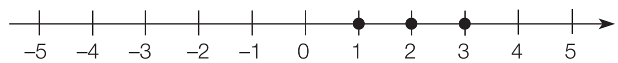
Graph the given inequality on the real number line.

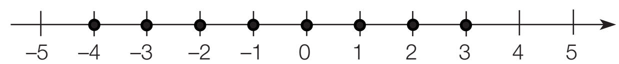
(a) $-5 < x < 4$, x a natural number

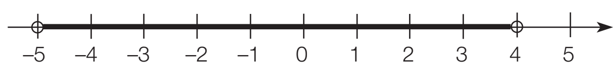
(b) $-5 < x < 4$, x an integer

(c) $-5 < x < 4$, x a real number

SOLUTIONS

(a) 

(b) 

(c) 

✔ Progress Check 8

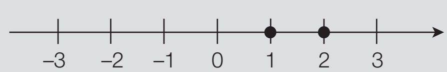
Graph the given inequality on the real number line.

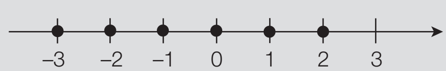
(a) $-3 \leqslant x \leqslant 2$, x a natural number

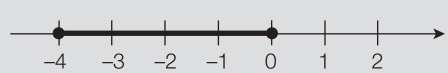
(b) $-2 \leqslant x < 3$, x an integer

(c) $-4 \leqslant x \leqslant 0$, x a real number

Answers

(a) 

(b) 

(c) 

It is sometimes convenient to use **set-builder notation** as a way of writing statements such as "A is the set of integers between -3 and 2." If we let I represent the set of integers, we write

$$A = \{x \in I \mid -3 < x < 2\}$$

Each part of this symbolic expression has an explicit meaning.

$\{\quad\}$	*read*	the set of all
$x \in I$	*read*	integers x
$\mid$	*read*	such that
$-3 < x < 2$	*read*	x is between -3 and 2

EXAMPLE 9 WORKING WITH SET-BUILDER NOTATION

If N is the set of natural numbers, write the statement "the set A of all natural numbers less than 10" in set-builder notation. List the elements of A.

SOLUTION

$A = \{x \in N \mid x < 10\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

✔ Progress Check 9

If N is the set of natural numbers, write the statement "A is the set of all odd natural numbers less than 12" in set-builder notation. List the elements of A.

Answer

$A = \{x \in N \mid x < 12 \text{ and } x \text{ is odd}\} = \{1, 3, 5, 7, 9, 11\}$

Exercise Set 1.6

In Exercises 1–24, find the value of the given expression.

1. $|2|$
2. $\left|-\frac{2}{3}\right|$
3. $|1.5|$
4. $|-0.8|$
5. $-|2|$
6. $-\left|-\frac{2}{5}\right|$
7. $|2-3|$
8. $|2-2|$
9. $|2-(-2)|$
10. $|2| + |-3|$

11. 12.

13. 14.

15. $|x| - |y|$ when $x = -1, y = -2$
16. $|x| - |x \cdot y|$ when $x = -3, y = 4$
17. $|x + y| + |x - y|$ when $x = -3, y = 2$
18. when $a = 1, b = 2$
19. when $x = -3, y = 4$
20. when $a = -3, b = 2$
21. when $a = 1, b = 3, c = 1$
22. when $a = -2, b = 3, c = -5$
23. when $a = 1.69, b = -7.43$, $c = 2.98$
24. $\dfrac{|-b|c-a||}{c|b-a|}$ when $a = 12.44$, $b = 4.74$, $c = -5.83$

In Exercises 25–30 the coordinates of points A and B are given. Find $\overline{AB}$.

25. 2, 5
26. -3, 6
27. -3, -1
28. -4, $\frac{11}{2}$
29. $-\frac{4}{5}$, $\frac{4}{5}$
30. 2, 2

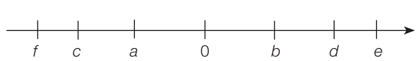
In Exercises 31–36, write each given statement using the symbols $<, >, \leqslant, \geqslant$.

31. 4 is greater than 1.
32. -2 is less than -1.
33. 2 is not greater than 3.
34. 3 is not less than 1.
35. 3 is nonnegative.
36. -2 is nonpositive.

In Exercises 37–51 make a true statement by replacing the square with the symbol $<$ or $>$.

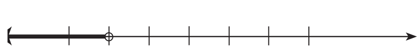
37. $3 \square 5$
38. $8 \square 2$
39. $4 \square -3$
40. $4 \square -6$
41. $-3 \square -2$
42. $-5 \square -4$
43. $-\frac{1}{2} \square \frac{1}{3}$
44. $\frac{1}{2} \square -\frac{1}{4}$
45. $-\frac{1}{5} \square -\frac{1}{3}$
46. $|-3| \square |5|$
47. $-|3| \square |4|$
48. $|-4| \square |-3|$
49. $|-2| \square 1$
50. $|4| \square 0$
51. $-|4| \square 0$

In Exercises 52–57 replace the square with the symbol $<$ or $>$ to make a true statement, referring to the number line below.

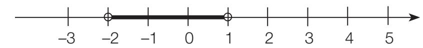


52. $a \square 0$
53. $b \square a$
54. $e \square f$
55. $d \square c$
56. $0 \square e$
57. $d \square a$

In Exercises 58–61 state the inequality represented on the given number line.

58. 

59. 

60. 

61. 

In Exercises 62–73 graph the inequality.

62. $x \leqslant -2$
63. $x \geqslant -3$
64. $x < 4$
65. $x > 1$
66. $-3 \leqslant x \leqslant 2$
67. $-4 < x < -2$
68. $-2 < x < 0$
69. $1 < x < 3$
70. $-2 < x \leqslant 3$, x an integer
71. $1 < x < 5$, x a natural number
72. $-3 \leqslant x \leqslant -1$, x an integer
73. $-3 \leqslant x < 2$, x a natural number

In Exercises 74–77 I is the set of integers, N is the set of natural numbers, and R is the set of real numbers. Write each statement in set-builder notation.

74. The set of natural numbers between -3 and 4, including -3
75. The set of integers between 2 and 4, including 4
76. The set of negative integers
77. The set of even natural numbers less than 6

In Exercises 78–83 list the elements of the given set. I is the set of integers, and N is the set of natural numbers.

78. $\{x \in I \mid -5 < x \leqslant -1\}$
79. $\{x \in I \mid -4 \leqslant x < 0\}$
80. $\{x \in N \mid -2 \leqslant x < 4\}$
81. $\{x \in N \mid 0 \leqslant x \leqslant 6\}$
82. $\{x \in I \mid -4 \leqslant x \leqslant 10$ and x is even$\}$
83. $\{x \in N \mid 1 < x \leqslant 6$ and x is odd$\}$

*84. Using the indicated values for a and b, verify the following properties of absolute value.

(a) $|a| \geqslant 0$ $\quad a = 2; a = -4$

(b) $|a| = |-a|$ $\quad a = -5; a = 3; a = 0$

(c) $|a - b| = |b - a|$ $\quad a = 3, b = 1; a = -2, b = -1$

(d) $|ab| = |a|\,|b|$ $\quad a = -2, b = -4$

(e) $\left| \quad \right| =$ $\quad a = 3, b = -5$

*85. Let a and b be the coordinates of two distinct points on the real number line. What does property (c) in the previous exercise say about measuring the distance between the two points?

*86. Verify that

$$|a + b| \leq |a| + |b|$$

using the following values of a and b:

$$a = 3, b = 2; a = -3, b = 2;$$
$$a = 3, b = -2; a = -3, b = -2$$

Chapter Summary

Terms and Symbols

$<, >, \leq, \geq$	40	equivalent fraction	12	percent, %	17
$\in, \notin$	3	exponent	28	prime number	15
absolute value, $\lvert\ \rvert$	36	factor	9	product	9
algebraic expression	21	fraction	10	quotient	9
associative properties	31	graph of an inequality	41	rational number	4
base	28	hierarchy of operations	22	real number line	5
cancellation principle	13	identity properties	34	real number system	5
closure properties	30	inequality symbols	40	reciprocal	11
common factor	13	integer	4	reduced form	13
commutative properties	31	inverse properties	34	set	14
constant	21	irrational number	5	set-builder notation	44
denominator	10	LCD	14	set notation	3
distributive properties	33	member	3	solution set	41
dividend	9	natural number	4	subset	3
divisor	9	numerator	10	variable	21
element	3	origin	5		

Key Ideas for Review

Topic	Page	Key Idea
Fractions	8	
multiplication denominators.	10	Find the product of the numerators and then divide by the product of the
division	11	Multiply the numerator by the reciprocal of the denominator.
addition and subtraction	14	If the denominators are the same, add or subtract the numerators and keep the common denominator. If the denominators are different, find the LCD and convert each fraction to an equivalent with the LCD as the denominator.
Percent	17	Percent is a fraction whose denominator is 100.

Conversions	17	Fractions, decimals, and percents are different ways of writing the same thing; you can always convert any form to any other form.
Rational number	4	A rational number is a quotient of two integers, p/q, with $q \neq 0$. The decimal form of a rational number either terminates or forms a repeating pattern.
Irrational number	5	An irrational number cannot be written as a quotient of two integers; the decimal equivalent never forms a repeating pattern.
Real number system	3	The real number system consists of the rational numbers and the irrational numbers.
Hierarchy of operations	22	Operations within parentheses are done *before* multiplication and division; addition and subtraction are done last.
Properties of the real numbers	33	See Table 2.
Absolute value	35	Absolute value is defined by $$\|a\| = \begin{cases} a & \text{when} \quad a \geqslant 0 \\ -a & \text{when} \quad a < 0 \end{cases}$$
Distance from origin	35	Absolute value represents distance and is always nonnegative. The distance from the origin to a point whose coordinate is a on the real number line is $\|a\|$.
Distance between points	37	The distance between points A and B, whose coordinates on the real number line are a and b, respectively, is $$\overline{AB} = \|b - a\| = \|a - b\|$$

Common Errors

1. $2x - y \neq 2(x - y)$. *Don't* assume grouping where it isn't indicated.

2. $3(a - b) = 3a - 3b$. *Don't* write $3(a - b) = 3a - b$.

3. $(-2)(-3)(-4) = -24$, not $+24$. When the number of negative factors is odd, the product is negative.

4. To evaluate an expression such as
$$\frac{3x + 2y}{x + 3y} \quad \text{when } x = 2 \text{ and } y = 1$$
work independently on the numerator and denominator before dividing.
$$\frac{3(2) + 2(1)}{2 + 3(1)} = \frac{6 + 2}{2 + 3} = \frac{8}{5}$$
Don't write
$$\frac{3(2) + 2(1)}{2 + 3(1)} = \frac{3 + 1}{1 + 3} = \frac{4}{4} = 1$$

5. The absolute value bars act as grouping symbols. We must work *inside* these grouping symbols before we can remove them.

6. The number π is irrational; therefore, we cannot say $\pi = \frac{22}{7}$ or $\pi = 3.14$. These are approximations for computational use only. We write $\pi \approx 3.14$ where the symbol $\approx$ is read as "is approximately equal to."

Review Exercises

Solutions to exercises whose numbers are in bold are in the Solutions Section in the back of the book.

1.1 In Exercises 1–3 write each set by listing its elements within braces.

1. The set of natural numbers between -5 and 4, inclusive.

2. The set of integers between -3 and -1, inclusive.

3. The subset of $x \in S$, $S = \{0.5, 1, 1.5, 2\}$, such that x is an even integer.

In Exercises 4–7 determine whether the statement is true (T) or false (F).

4. $\sqrt{7}$ is a real number.

5. -35 is a natural number.

6. -14 is not an integer.

7. 0 is an irrational number.

8. Draw a real number line and plot the following points.
(a) 3 (b) -5 (c) $\frac{1}{2}$ (d) -1.5

In Exercises 9–11 determine which of the two numbers appears second when viewed from left to right on the real number line.

9. 3, 2 10. $-4, -5$ 11. $0, -2$

1.2 In Exercises 12–17 perform the indicated operations.

12. $\dfrac{(2+3)4}{10} + 4 \cdot 3$ 13. $\dfrac{(3-5)(4-16)}{(3+1)(-2)} + \dfrac{1}{2}$

14. $\dfrac{2}{3} \cdot \dfrac{4}{5} \cdot \dfrac{6}{7}$ 15. $\dfrac{3}{4} \div \dfrac{5}{8}$

16. $\dfrac{1 + \frac{1}{2}}{\frac{3}{4} + \frac{1}{2}}$ 17. $\dfrac{\frac{2}{3} + \frac{1}{6}}{\frac{2}{9} - \frac{3}{2}}$

In Exercises 18 and 19 change the given percent to both fractional and decimal forms.

18. 7% **19.** 2.25%

In Exercises 20 and 21 convert the given number to percent.

20. 4.52 21. 0.021

22. Suppose that your school tax bill reads: "$800 in taxes due Nov 1, reduced to $784 if paid by Oct 15." What is the percent of the discount?

1.3 In Exercises 23–26 determine whether the given statement is true (T) or false (F).

23. $2x + 4 = 10$ when $x = 3$

24. $3x - 2 = 6$ when $x = 3$

25. $3x - 4y = 6$ when $x = 1, y = 2$

26. $2x + 5y = 11$ when $x = -2, y = 3$

27. A salesperson receives $3.25x + 0.15y$ dollars, where x is the number of hours worked and y is the number of miles of automobile usage. Find the amount due the salesperson if $x = 12$ hours and
$y = 80$ miles.

1.4 In Exercises 28–33 simplify.

28. $3 + (-5)$ 29. $6 - 8$

30. $(-5) + (-3)$ 31. $(-3) - (-2)$

32. $(-2)\left(-\dfrac{1}{2}\right)$ 33. $\dfrac{-16}{-2}$

34. Evaluate $x - 3y$ when $x = 2, y = -3$.

35. A stereo shop had the following financial history during its first three years of operation. It lost x dollars during the first year and made a profit of y dollars during the second year. Its profit during the third year was $1000. Write an expression for the net profit or loss after three years.

1.5 In Exercises 36–39 identify the property (or properties) that justifies the given statement.

36. $(3 + 4)x = 3x + 4x$

37. $a + (b + c) = c + (a + b)$

38. $c(a + b) = bc + ac$

39. $3(ab) = b(3a)$

In Exercises 40–43 find and correct the mistake.

40. $2(a + 3) = 2a + 3$

41. $\dfrac{4 + a}{2} = 2 + a$

42. $-2(a - 3) = -2a - 6$

43. $2(ab) = (2b)(2a)$

1.6 In Exercises 44 and 45 find the value of the expression.

44. $\dfrac{|3| - |4|}{|2| + |-5|}$

45. $\dfrac{|2 - 2b| + |a - b|}{|ab|}$ when $a = -2$, $b = 3$

46. Provide a counterexample for the following statement:

If a and b are real numbers such that $|a| = |b|$, then $a = b$.

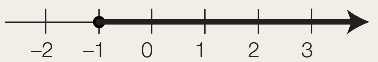
In Exercises 47 and 48 the coordinates of points A and B are given. Find $\overline{AB}$.

47. $-3, 2$

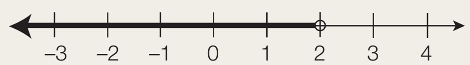
48. $-4, -8$

In Exercises 49 and 50 state the inequality represented on the given number line.

49.



50.



In Exercises 51 and 52 graph the inequality on the real number line.

51. $x > -2$

52. $-2 \leqslant x < 5$

Progress Test 1A

1. The numbers $3, -\frac{2}{3}$, and 0.72 are all
   (a) natural numbers (b) rational numbers
   (c) irrational numbers (d) none of these

2. The numbers $-\frac{\pi}{2}, \sqrt{3}$, and $-\frac{4}{5}$ are all
   (a) irrational numbers (b) rational numbers
   (c) real numbers (d) none of these

3. On a real number line, indicate the integers that are greater than -3 and less than 4.

4. Evaluate $\dfrac{3a - 4b}{2a - b}$ when $a = 3$, $b = 2$.

5. Evaluate $\dfrac{2x - 6y}{x - y}$ when $x = -1$, $y = 1$.

6. Evaluate $\dfrac{3[(a + 2b) - (2b - a)]}{c}$ when $a = -2$, $b = -1$, $c = -6$.

7. Simplify $5(x - y) - 3(2x - y)$.

8. Simplify $2\left(\dfrac{a + b}{4}\right) + \left(\dfrac{b - a}{2}\right)$.

9. Evaluate $\dfrac{|2 - 8|}{|2| + |-8|}$.

10. Evaluate $3|x| - 2|2y|$ when $x = -2$, $y = -3$.

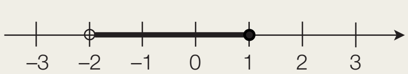
11. Evaluate $\left|\dfrac{-2|3x| + 3|-y|}{|x + y|}\right|$ when $x = -2$, $y = 1$.

12. Evaluate $|x| \cdot |y| - 2|x \cdot y|$ when $x = -1$, $y = 2$.

13. Graph the inequality $-1 \leqslant x < 3$.

14. Graph the inequality $-4 \leqslant x \leqslant 1$, x an integer.

15. Give the inequality represented on the following number line.



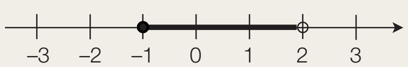
Progress Test 1B

1. The numbers -2, 0.45, and $\frac{7}{9}$ are all
   (a) integers (b) irrational numbers
   (c) rational numbers (d) none of these
2. The numbers $-\sqrt{7}$, 2π, and -0.49 are all
   (a) natural numbers (b) irrational numbers
   (c) real numbers (d) none of these
3. On a real number line, indicate the integers that are greater than -5 and less than -1.
4. Evaluate $\dfrac{2m - 5n}{3m - n}$ when $m = 2$, $n = 4$.
5. Evaluate $\dfrac{x + 2y}{2x - y}$ when $x = 3$, $y = -5$.
6. Evaluate $\dfrac{-2[(p-2q)-2r-p]}{p \times q}$ when $p = 2$,

   $q = -3$,

   $r = \frac{1}{2}$.
7. Simplify $7(x + 2y) - 2(3x - y)$.
8. Simplify $3\left(\dfrac{a - 2b}{6}\right) - \left(\dfrac{b - 2a}{2}\right)$.
9. Evaluate $\dfrac{|-3| - |4 - 7|}{|-4| + |-2|}$.
10. Evaluate $\dfrac{|x|}{2} - 3|3y|$ when $x = -4$, $y = -5$.
11. Evaluate $\dfrac{3|2y| - 3|x|}{-|x - 2y|}$ when $x = -2$, $y = 1$.
12. Evaluate $\left|\dfrac{x}{3} \cdot \dfrac{y}{2}\right| - 3|2x \cdot y|$ when $x = -6$,

    $y = 4$.
13. Graph the inequality $-2 < x \leq 5$.
14. Graph the inequality $-5 \leq x \leq -1$, x an integer.
15. Give the inequality represented on the following number line.



Chapter 1 Project

To study the nature and impact of hurricanes, from a scientific as well as a human standpoint, it is necessary to use and understand real numbers. In the chapter opener, you read about the Saffir-Simpson Hurricane Scale. What subset of the real numbers does this scale use?

Look at Exercise 42 in Section 1.1, and Exercises 81 and 82 in Section 1.2. Now do some additional research on hurricanes. Use percents, inequalities, and number lines to make a report on the data you discover. For example, what percentage of deaths due to weather were caused by hurricanes in your area last year? What range of wind speeds are associated with a category 3 hurricane?