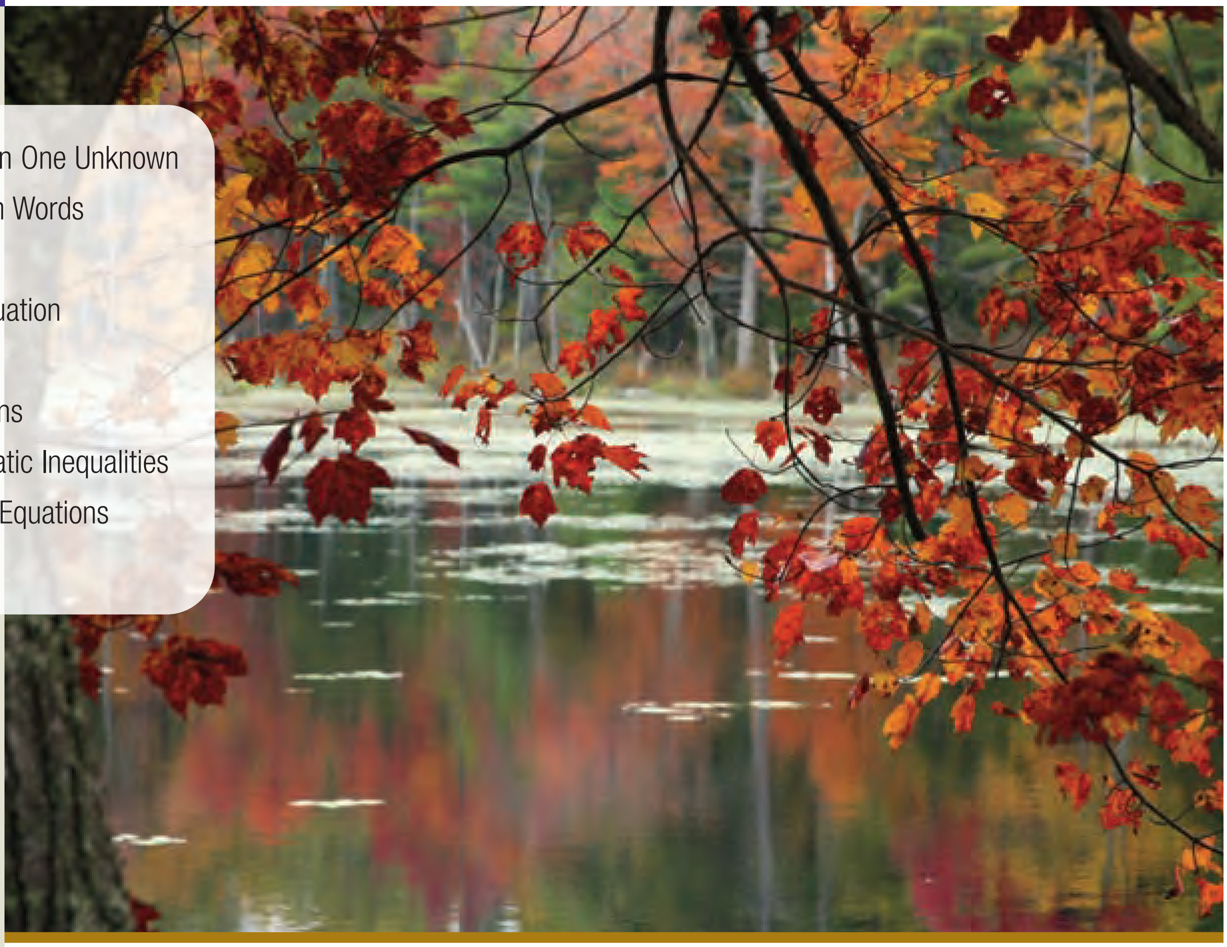
2 Equations and Inequalities



The Internet is a short form of the word "internetworking." The Internet is a vast data network, with humble origins in the 1960s. A network, whether it connects computers or people, is just a way of facilitating communication. In a **full-mesh network**, elements are linked pairwise—that is, any two elements in the system are linked directly, without intermediary.

How many elements (users) could be linked in a full-mesh network with 190 two-way links? The answer to this problem is found by solving a quadratic equation (see the Chapter Project). This chapter will show you how.

Explore the Internet for its many mathematical offerings! Check out a site which is organized according to the Mathematics Subject Classification created by the American Mathematical Society. Look up graph theory to learn more about networking.

A major concern of algebra is the solution of equations. Does a given equation have a solution? Is it possible for an equation to have more than one solution? Is there a procedure for solving an equation? In this chapter we will explore the answers to these questions for polynomial equations of the first and second degree. We will also see that the ability to solve equations enables us to tackle a wide variety of applications and word problems.

Linear inequalities also play an important role in solving word problems. For example, if we are required to combine food products in such a way that a specified minimum daily requirement for various nutrients is provided, we need to use inequalities. Many important industries, including steel and petroleum, use computers daily to solve problems that involve thousands of inequalities. The solutions to such problems enable a company to optimize its "product mix" and its profitability.

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2.1 Linear Equations in One Unknown

Solving Equations

Expressions of the form

$$x - 2 = 0 \qquad x^2 - 9 = 0 \qquad 3(2x - 5) = 3$$

$$2x + 5 = \sqrt{x - 7} \qquad \frac{1}{2x + 3} = 5 \qquad x^3 - 3x^2 = 32$$

are examples of equations in the unknown x. An **equation** states that two algebraic expressions are equal. We refer to these expressions as the **left-hand side** and the **right-hand side** of the equation.

Our task is to find values of the unknown for which the equation is satisfied. These values are called **solutions** or **roots** of the equation, and the set of all solutions is called the **solution set**. For example, 2 is a solution of the equation $3x - 1 = 5$ since $3(2) - 1 = 5$. However, -2 is *not* a solution since $3(-2)-1 \neq 5$.

Equations that do not have solutions in one number system may have solutions in a larger number system. For example, the equation $2x - 5 = 0$ has no integer solutions but does have a solution among the rational numbers, namely $\frac{5}{2}$. Similarly, the equation $x^2 = -4$ has no solutions among the real numbers but does have solutions if we consider complex numbers, namely $2i$ and $-2i$. The solution sets of these two equations are $\{\frac{5}{2}\}$ and $\{2i, -2i\}$, respectively.

Identities and Conditional Equations

We say that an equation is an **identity** if it is true for every real number for which both sides of the equation are defined. For example, the equation

$$x^2 - 1 = (x + 1)(x - 1)$$

is an identity because it is true for all real numbers. (Try any number and check that this equation holds.) The equation

$$x - 5 = 3$$

is only true when $x = 8$. (Try any number not equal to 8 and check that this equation does not hold.) An equation such as $x - 5 = 3$, which is not true for all values of x, is called a **conditional equation**.

When we say that we want to "solve an equation," we mean that we want to find *all* solutions or roots. If we can replace an equation with another, simpler equation that has the same solutions, we will have an approach to solving equations. Equations having the same solutions are called **equivalent equations**. For example, $3x - 1 = 5$ and $3x = 6$ are equivalent equations because it can be shown that $\{2\}$ is the solution set of both equations.

There are two important rules that allow us to replace an equation with an equivalent equation.

Equivalent Equations

The solutions of a given equation are not affected by the following operations:

1. addition (or subtraction) of the same number or expression on both sides of the equation
2. multiplication (or division) by the same number, different from 0, on both sides of the equation

EXAMPLE 1 SOLVING EQUATIONS

Solve $3x + 4 = 13$.

SOLUTION

We apply the preceding rules to this equation. The strategy is to isolate x, so we *subtract 4 from both sides of the equation.*

$$3x + 4 - 4 = 13 - 4$$
$$3x = 9$$

Dividing both sides by 3, we obtain the solution

$$x = 3$$

We check by substitution to make sure that 3 does, indeed, satisfy the original equation.

$$\begin{aligned} \text{left-hand side} &= 3x + 4 \qquad \text{right-hand side} = 13 \\ &= 3(3) + 4 \\ &= 13 \end{aligned}$$

Although $x = 3$ is an equation that is *equivalent* to the original equation, in common usage we say that $3x + 4 = 13$ "has the solution $x = 3$."

When the given equation contains rational expressions, we eliminate fractions by first multiplying by the least common denominator of all fractions present. This technique is illustrated in Examples 2, 3, and 4.

EXAMPLE 2 SOLVING EQUATIONS

Solve the equation.

$$\frac{5}{6}x - \frac{4}{3} = \frac{3}{5}x + 1$$

SOLUTION

We first eliminate fractions by multiplying both sides of the equation by the LCD of all fractions, which is 30.

$$\left(\frac{5}{6}x - \frac{4}{3}\right)(30) = \left(\frac{3}{5}x + 1\right)(30)$$

$$25x - 40 = 18x + 30$$

$$7x = 70$$

$$x = 10$$

Verify that $x = 10$ is a solution of the original equation.

✔ Progress Check

Solve and check.

a. $-\frac{2}{3}(x - 5) = \frac{3}{2}(x + 1)$ b. $\frac{1}{3}x + 2 - 3\left(\frac{x}{2} + 4\right) = 2\left(\frac{x}{4} - 1\right)$

Answers

a. $\frac{11}{13}$ b. $-\frac{24}{5}$

Solving Linear Equations

The equations we have solved are all of the first degree and involve only one unknown. Such equations are called **first-degree equations in one unknown**, or more simply, **linear equations**. The general form of such equations is

$$ax + b = 0$$

where a and b are any real numbers and $a \neq 0$. Let us see how to solve this equation.

$$\begin{aligned} ax + b &= 0 \\ ax + b - b &= 0 - \mathrm{b} && \text{Subtract } b \text{ from both sides.} \\ ax &= -b \\ \frac{ax}{a} &= \frac{-b}{a} && \text{Divide both sides by } a \neq 0. \\ x &= -\frac{b}{a} \end{aligned}$$

We verify that this is a solution.

$$a\left(-\frac{b}{a}\right) + b = 0$$

Furthermore, it can be shown that this is the only solution. We have thus obtained the following result:

Roots of a Linear Equation

The linear equation $ax + b = 0$, $a \neq 0$, has exactly one solution:

$$x = -\frac{b}{a}$$

Sometimes we are led to linear equations in the course of solving other equations. The following example illustrates this situation.

EXAMPLE 3 SOLVING EQUATIONS

Solve.

$$\frac{5x}{x+3} - 3 = \frac{1}{x+3}$$

SOLUTION

The LCD of all fractions is $x + 3$. Multiplying both sides of the equation by $x + 3$ to eliminate fractions, we obtain

$$\begin{aligned} 5x - 3(x + 3) &= 1 \\ 5x - 3x - 9 &= 1 \\ 2x &= 10 \\ x &= 5 \end{aligned}$$

Checking the solution, we have

$$\text{left-hand side} = \frac{5x}{x+3} - 3 = \frac{5(5)}{5+3} - 3 = \frac{25}{8} - 3 = \frac{25}{8} - \frac{24}{8} = \frac{1}{8}$$

$$\text{right-hand side} = \frac{1}{x+3} = \frac{1}{5+3} = \frac{1}{8}$$

We said earlier that multiplication (or division) of both sides of an equation by any nonzero number results in an equivalent equation. What happens if we multiply or divide an equation by an expression that contains an unknown? In Example 3, this procedure worked and gave us a solution. However, this may not always be so since the answer we obtain may produce a zero denominator when substituted back into the original equation. Therefore, the following rule must be carefully observed:

Multiplying by an Unknown

Multiplication (or division) by the same expression on both sides of an equation may result in an equation that is *not* equivalent to the original equation. Always verify that the answer obtained to the subsequent equation is, indeed, a solution to the original equation.

EXAMPLE 4 EQUATIONS WITH NO SOLUTION

Solve and check.

$$\frac{8x+1}{x-2} + 4 = \frac{7x+3}{x-2}$$

SOLUTION

The LCD of all fractions is $x - 2$. Multiplying both sides of the equation by $x - 2$, we eliminate fractions and obtain

$$8x + 1 + 4(x - 2) = 7x + 3$$

$$8x + 1 + 4x - 8 = 7x + 3$$

$$5x = 10$$

$$x = 2$$

Checking our answer, we find that $x = 2$ is not a solution since substituting $x = 2$ in the original equation yields a denominator of zero. We conclude that the given equation has no solution.

✔ Progress Check

Solve and check.

a. $\dfrac{3}{x} - 1 = \dfrac{1}{2} - \dfrac{6}{x}$ b. $-\dfrac{2x}{x+1} = 1 + \dfrac{2}{x+1}$

Answers

a. $x = 6$ b. no solution

EXAMPLE 5 EQUATIONS WITH NO SOLUTION

Solve the equation $2x + 1 = 2x - 3$.

SOLUTION

Subtracting $2x$ from both sides, we have

$$2x + 1 - 2x = 2x - 3 - 2x$$
$$1 = -3$$

This equivalent equation is a contradiction, so we conclude that the given equation has no solution.

Exercise Set 2.1

In Exercises 1–4, determine whether the given statement is true (T) or false (F).

1. $x = -5$ is a solution of $2x + 3 = -7$.
2. $x = \dfrac{5}{2}$ is a solution of $3x - 4 = \dfrac{5}{2}$.
3. $x = \dfrac{6}{4-k}$, $k \neq 4$
   is a solution of $kx + 6 = 4x$.
4. $x = \dfrac{7}{3k}$, $k \neq 0$
   is a solution of $2kx + 7 = 5x$.

In Exercises 5–24, solve the given linear equation and check your answer.

5. $3x + 5 = -1$
6. $5r + 10 = 0$
7. $2 = 3x + 4$
8. $\dfrac{1}{2}s + 2 = 4$
9. $\dfrac{3}{2}t - 2 = 7$
10. $-1 = -\dfrac{2}{3}x + 1$
11. $0 = -\dfrac{1}{2}a - \dfrac{2}{3}$
12. $4r + 4 = 3r - 2$
13. $-5x + 8 = 3x - 4$
14. $2x - 1 = 3x + 2$
15. $-2x + 6 = -5x - 4$
16. $6x + 4 = -3x - 5$
17. $2(3b + 1) = 3b - 4$

18. $-3(2x + 1) = -8x + 1$

19. $4(x - 1) = 2(x + 3)$

20. $-3(x - 2) = 2(x + 4)$

21. $2(x + 4) - 1 = 0$

22. $3a + 2 - 2(a - 1) = 3(2a + 3)$

23. $-4(2x + 1) - (x - 2) = -11$

24. $3(a + 2) - 2(a - 3) = 0$

Solve for x in Exercises 25–28.

25. $kx + 8 = 5x$

26. $8 - 2kx = -3x$

27. $2 - k + 5(x - 1) = 3$

28. $3(2 + 3k) + 4(x - 2) = 5$

Solve and check in Exercises 29–44.

29. $\dfrac{x}{2} = \dfrac{5}{3}$

30. $\dfrac{3x}{4} - 5 = \dfrac{1}{4}$

31. $\dfrac{2}{x} + 1 = \dfrac{3}{x}$

32. $\dfrac{5}{a} - \dfrac{3}{2} = \dfrac{1}{4}$

33. $\dfrac{2y - 3}{y + 3} = \dfrac{5}{7}$

34. $\dfrac{1 - 4x}{1 - 2x} = \dfrac{9}{8}$

35. $\dfrac{1}{x - 2} + \dfrac{1}{2} = \dfrac{2}{x - 2}$

36. $\dfrac{4}{x - 4} - 2 = \dfrac{1}{x - 4}$

37. $\dfrac{2}{x - 2} + \dfrac{2}{x^2 - 4} = \dfrac{3}{x + 2}$

38. $\dfrac{3}{x - 1} + \dfrac{2}{x + 1} = \dfrac{5}{x^2 - 1}$

39. $\dfrac{x}{x - 1} - 1 = \dfrac{3}{x + 1}$

40. $\dfrac{2}{x - 2} + 1 = \dfrac{x + 2}{x - 2}$

41. $\dfrac{4}{b} - \dfrac{1}{b + 3} = \dfrac{3b + 2}{b^2 + 2b - 3}$

42. $\dfrac{3}{x^2 - 2x} + \dfrac{2x - 1}{x^2 + 2x - 8} = \dfrac{2}{x + 4}$

43. $\dfrac{3r + 1}{r + 3} + 2 = \dfrac{5r - 2}{r + 3}$

44. $\dfrac{2x - 1}{x - 5} + 3 = \dfrac{3x - 2}{5 - x}$

In Exercises 45–48, indicate whether the equation is an identity (I) or a conditional equation (C).

45. $x^2 + x - 2 = (x + 2)(x - 1)$

46. $(x - 2)^2 = x^2 - 4x + 4$

47. $2x + 1 = 3x - 1$

48. $3x - 5 = 4x - x - 2 - 3$

In Exercises 49–54, write (T) if the equations within each exercise are all equivalent equations and (F) if they are not equivalent.

49. $2x - 3 = 5$ $\quad 2x = 8$ $\quad x = 4$

50. $5(x - 1) = 10$ $\quad x - 1 = 2$ $\quad x = 3$

51. $x(x - 1) = 5x$ $\quad x - 1 = 5$ $\quad x = 6$

52. $x = 5$ $\quad x^2 = 25$

53. $3(x^2 + 2x + 1) = -6$ $\quad x^2 + 2x + 1 = -2$
$(x + 1)^2 = -2$

54. $(x + 3)(x - 1) = x^2 - 2x + 1$
$(x + 3)(x - 1) = (x - 1)^2$ $\quad x + 3 = x - 1$

55. Write repeating decimal fractions as rational equivalents.

Example:

$$\begin{aligned} N &= 0.1515\ldots = 0.\overline{15} \\ 100N &= 15.\overline{15} \\ -N &= -0.\overline{15} \\ \hline 99\,N &= 15 \\ N &= \frac{15}{99} = \frac{5}{33} \end{aligned}$$

a. $0.\overline{2}$ $\qquad$ b. $0.\overline{123}$

c. $1.\overline{35}$ $\qquad$ d. $0.\overline{9}$

56. Find the error in the following argument. Assume that $a = b \neq 0$.

$$\begin{aligned} a^2 &= ab \\ a^2 - b^2 &= ab - b^2 \\ (a + b)(a - b) &= b(a - b) \\ a + b &= b \\ b + b &= b \text{ (since } a = b) \\ 2b &= b \\ 2 &= 1 \text{ (since } b \neq 0) \end{aligned}$$

57. Solve

a. $\dfrac{w - c}{w - d} = \dfrac{c^2}{d^2}$ $\qquad$ for w

b. $a^2 = \dfrac{a + c}{x} + c^2$ $\qquad$ for x

c. $(a - y)(y + b) - c(y + c) = (c - y)(y + c) + ab \quad$ for y

58. Solve for y.

a. $y + \dfrac{c}{y-3} = 3 + \dfrac{c}{y-3}$

b. $y + \dfrac{c}{y-3} = -3 + \dfrac{c}{y-3}$

59. The golden ratio is given by

$$T = \frac{1 + \sqrt{5}}{2}$$

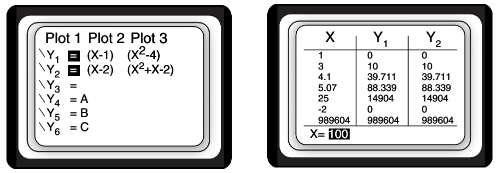
(See Chapter 1, Review Exercise 78.) Show that

$$T = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{T}}}$$

60. Determine if the equation is an identity experimentally by setting up a TABLE in your graphing calculator and entering a variety of values for x.

$$(x^2 + 5x + 6)(x - 3) = (x^2 - x - 6)(x + 3)$$

Example for Exercise 60:



61. *Mathematics in Writing:* Set up a table like the one in Exercise 60, but pick a Y_1 and Y_2 so that $Y_1 = Y_2$ is a conditional equation. Explain in a brief paragraph how the tables differ and what the difference tells you about the equations.

2.2 Applications: From Words to Algebra

Many applied problems lead to linear equations. The challenge of applied problems is translating words into appropriate algebraic forms.

The steps listed here can guide you in solving word problems.

Step 1. Read the problem through the first time to get a general idea of what is being asked.

Step 2. Read the problem a second time to recognize what may be important in determining that which is to be found.

Step 3. If possible, estimate the solution to this problem, and then compare this estimate with your final answer.

Step 4. Let some algebraic symbol denote the quantity to be found.

Step 5. If possible, represent other quantities in the problem in terms of the algebraic symbol designated in *Step 4*.

Step 6. Find various relationships (equations or inequalities) in the problem.

Step 7. Use relationships established in *Step 6* to find the solution to the problem.

Step 8. Verify that your answer is, indeed, the solution to the problem.

The words and phrases in Table 1 may prove helpful in translating a word problem into an algebraic expression that can be solved.

EXAMPLE 1 PRICES AND DISCOUNTS

If you pay \$66 for a car radio after receiving a 25% discount, what was the price of the radio before the discount?

SOLUTION

Let p = the price of the radio (in dollars) before the discount. Then

$$0.25p = \text{the amount discounted}$$

and the price of the radio after the discount is given by

$$p - 0.25p$$

Hence

$$\begin{aligned} p - 0.25p &= 66 \\ 0.75p &= 66 \\ p &= \frac{66}{0.75} = 88 \end{aligned}$$

The price of the radio was \$88 before the discount.

TABLE 1 Translation of Words into Algebraic Expressions

Word or Phrase	Algebraic Symbol	Example	Algebraic Expression
Sum	$+$	Sum of two numbers	$a + b$
Difference	$-$	Difference of two numbers	$a - b$
		Difference of a number and 3	$x - 3$
Product	$\times$ or $\cdot$	Product of two numbers	$a \cdot b$, $(a)(b)$, or ab
Quotient	$\div$ or /	Quotient of two numbers	$\frac{a}{b}$, a/b, or $a \div b$
Exceeds		a exceeds b by 3	$a = b + 3$
More than		a is 3 more than b	or
More of		There are 3 more of a than of b.	$a - 3 = b$
Twice		Twice a number	$2x$
		Twice the difference of x and 3	$2(x - 3)$
		3 more than twice a number	$2x + 3$
		3 less than twice a number	$2x - 3$
Is or equals	$=$	The sum of a number and 3 is 15.	$x + 3 = 15$

Coin Problems

When interpreting coin problems, always distinguish between the *number* of coins and the *value* of the coins. You may also find it helpful to use a chart, as in the following example:

EXAMPLE 2 COINS

A purse contains \$3.20 in quarters and dimes. If there are 3 more quarters than dimes, how many coins of each type are there?

SOLUTION

In this problem, we may let the unknown represent either the number of quarters or the number of dimes. Let

$$q = \text{the number of quarters}$$

Then

$$q - 3 = \text{the number of dimes}$$

since there are 3 more quarters than dimes.

Note that the number of coins times the value of each coin in cents is equal to the total value in cents using that particular coin.

	Number of coins ×	Value of each coin in cents	= Total value in cents using that coin
Quarters	q	25	$25q$
Dimes	$q - 3$	10	$10(q - 3)$

We know that

$$\text{total value} = (\text{value of quarters}) + (\text{value of dimes})$$

$$320 = 25q + 10(q - 3)$$
$$320 = 25q + 10q - 30$$
$$350 = 35q$$
$$10 = q$$

Then

$$q = \text{number of quarters} = 10$$
$$q - 3 = \text{number of dimes} = 7$$

Now verify that the total value of all the coins is \$3.20.

Simple Interest

Interest is the fee charged for borrowing money. In this section we will deal only with simple interest, which assumes the fee to be a fixed percentage r of the amount borrowed. We call the amount borrowed the **principal** and denote it by P.

If the principal P is borrowed at a simple annual interest rate r, then the interest due at the end of each year is Pr, and the total interest I due at the end of t years is

$$I = Prt$$

Consequently, if S is the total amount owed at the end of t years, then

$$S = P + I = P + Prt$$

since both the principal and interest are to be repaid. Thus, the basic formulas for simple interest calculations are

$$I = Prt$$
$$S = P + Prt$$

EXAMPLE 3 SIMPLE INTEREST

A part of \$7000 was borrowed at 6% simple annual interest and the remainder at 8%. If the total amount of interest due after 3 years is \$1380, how much was borrowed at each rate?

SOLUTION

Let

$$s = \text{the amount borrowed at } 6\%$$

Then

$$7000 - s = \text{the amount borrowed at } 8\%$$

since the total amount is \$7000. We can display the information in table form using the equation $I = Prt$.

	P	×	r	×	t	=	Interest
6% Portion	s		0.06		3		$0.18s$
8% Portion	$7000 - s$		0.08		3		$0.24(7000 - s)$

Note that we write the rate r in its decimal form, so that $6\% = 0.06$ and $8\% = 0.08$.

Since the total interest of \$1380 is the sum of the interest from the two portions, we have

$$1380 = 0.18s + 0.24(7000 - s)$$
$$1380 = 0.18s + 1680 - 0.24s$$
$$0.06s = 300$$
$$s = 5000$$

We conclude that \$5000 was borrowed at 6% and \$2000 was borrowed at 8%.

Distance Problems (Uniform Motion)

Here is the key to the solution of distance problems.

$$\text{Distance} = (\text{Rate})(\text{Time})$$

or

$$d = r \cdot t$$

The relationships that permit you to write an equation are sometimes obscured by the words. Here are some questions to ask as you set up a distance problem.

1. Are there two distances that are equal? (Will two objects have traveled the same distance? Is the distance on a return trip the same as the distance going?)
2. Is the sum (or difference) of two distances equal to a constant? (When two objects are traveling toward each other, they meet when the sum of the distances traveled by them equals the original distance between them.)

EXAMPLE 4 TRAVEL

Two trains leave New York for Chicago. The first train travels at an average speed of 60 mph. The second train, which departs an hour later, travels at an average speed of 80 mph. How long will it take the second train to overtake the first train?

SOLUTION

Since we are interested in the time the second train travels, we let

t = the number of hours the second train travels

Then

$$t + 1 = \text{the number of hours the first train travels}$$

since the first train departs 1 hour earlier. We display the information in table form using the equation $d = rt$.

	Rate	x	Time	=	Distance
First train	60		$t + 1$		$60(t + 1)$
Second train	80		t		$80t$

At the moment the second train overtakes the first, they must both have traveled the *same* distance. Thus,

$$60(t + 1) = 80t$$
$$60t + 60 = 80t$$
$$60 = 20t$$
$$3 = t$$

It takes the second train 3 hours to catch up with the first train.

Mixture Problems

One type of mixture problem involves mixing varieties of a commodity, say two or more types of coffee, to obtain a mixture with a desired value. If the commodity is measured in pounds, the relationships we need are as follows:

$$\text{(Number of pounds)(Price per pound)} = \text{Value of commodity}$$
$$\text{Sum of weights of all varieties} = \text{Weight of mixture}$$
$$\text{Sum of values of all varieties} = \text{Value of mixture}$$

EXAMPLE 5 MIXTURES

How many pounds of Brazilian coffee worth \$10 per pound must be mixed with 20 pounds of Colombian coffee worth \$8 per pound to produce a mixture worth \$8.40 per pound?

SOLUTION

Let B = number of pounds of Brazilian coffee. We display all the information, using cents in place of dollars.

Type of coffee	Number of pounds	x	Price per pound	=	Value (in cents)
Brazilian	B		1000		$1000B$
Colombian	20		800		16,000
Mixture	$B + 20$		840		$840(B + 20)$

Note that the weight of the mixture equals the sum of the weights of the Brazilian and Colombian coffees that make up the mixture. Since the value of the mixture is the sum of the values of the two types of coffee,

$$\begin{aligned} \text{value of mixture} &= (\text{value of Brazilian}) + (\text{value of Colombian}) \\ 840(B + 20) &= 1000B + 16{,}000 \\ 840B + 16{,}800 &= 1000B + 16{,}000 \\ 800 &= 160B \\ 5 &= B \end{aligned}$$

We must add 5 pounds of Brazilian coffee to make the required mixture.

Work Problems

Work problems typically involve two or more people or machines working on the same task. The key to these problems is to express the *rate of work per unit of time,* whether an hour, a day, a week, or some other unit. For example, if a machine can do a job in 5 days, then

$$\text{rate of machine} = \frac{1}{5} \text{ job per day}$$

If this machine is used for 2 days, it performs $2(\frac{1}{5}) = \frac{2}{5}$ of the job. In summary:

If a machine (or person) can complete a job in n days, then

$$\text{Rate of machine (or person)} = \frac{1}{n} \text{ Job per day}$$
$$\text{Work done} = (\text{Rate})(\text{Time})$$

EXAMPLE 6 WORK

Using a small mower, at 12 noon a student begins to mow a lawn, a job that would take 9 hours working alone. At 1 P.M. another student, using a tractor, joins the first student and they complete the job together at 3 P.M. How many hours would it take to do the job using only the tractor?

SOLUTION

Let t = number of hours to do the job by tractor alone. The small mower works from 12 noon to 3 P.M., or 3 hours. The tractor is used from 1 P.M. to 3 P.M., or 2 hours.

All the information can be displayed in table form.

	Rate	×	Time	=	Work done
Small mower	$\frac{1}{9}$		3		$\frac{3}{9} = \frac{1}{3}$
Tractor	$\frac{1}{t}$		2		$\frac{2}{t}$

Since

$$\begin{array}{ccccc} \text{Work done by small mower} & + & \text{Work done by tractor} & = & 1 \text{ Whole job} \\ \frac{1}{3} & + & \frac{2}{t} & = & 1 \end{array}$$

To solve, multiply both sides by the LCD, which is $3t$.

$$\begin{aligned} \left(\frac{1}{3} + \frac{2}{t}\right)(3t) &= 1(3t) \\ t + 6 &= 3t \\ t &= 3 \end{aligned}$$

Thus, by tractor alone, the job can be done in 3 hours.

Formulas

The circumference C of a circle is given by the formula

$$C = 2\pi r$$

where r is the radius of the circle. For every value of r, the formula gives us a value of C. If $r = 20$, we have

$$C = 2\pi(20) = 40\pi$$

It is sometimes convenient to be able to turn a formula around, that is, to be able to solve for a different variable. For example, if we want to express the radius of a circle in terms of the circumference, we have

$$\begin{aligned} C &= 2\pi r \\ \frac{C}{2\pi} &= \frac{2\pi r}{2\pi} \qquad \text{Dividing by } 2\pi \\ \frac{C}{2\pi} &= r \end{aligned}$$

Now, given a value of C, we can determine a value of r.

EXAMPLE 7 MANIPULATION OF FORMULAS

If an amount P is borrowed at the simple annual interest rate r, then the amount S due at the end of t years is

$$S = P + Prt$$

Solve for P.

SOLUTION

$$P + Prt = S$$

$$P(1 + rt) = S \qquad \text{Common factor } P$$

$$P = \frac{S}{1 + rt} \qquad \text{Dividing both sides by } (1 + rt)$$

Graphing Calculator Alert

A formula may be stored as a PROGRAM in your graphing calculator. The program may be written merely to display the formula, or to ask the user for the input values and then evaluate the formula for those particular values. Many formulas are available online, or you could learn to write your own. Consult your owner's manual for details. The owner's manual may be available online.

Exercise Set 2.2

In Exercises 1–3, let n represent the unknown. Translate from words to an algebraic expression or equation.

1. The number of blue chips is 3 more than twice the number of red chips.
2. The number of station wagons on a parking lot is 20 fewer than 3 times the number of sedans.
3. Five less than 6 times a number is 26.

In Exercises 4–41, translate from words to an algebraic problem and solve.

4. Janis is 3 years older than her sister. Thirty years from now the sum of their ages will be 111. Find the current ages of the sisters.
5. John is presently 12 years older than Fred. Four years ago John was twice as old as Fred. How old is each now?
6. The larger of two numbers is 3 more than twice the smaller. If their sum is 18, find the numbers.
7. Find three consecutive integers whose sum is 21.
8. A certain number is 5 less than another number. If their sum is 11, find the two numbers.
9. A resort guarantees that the average temperature over the period Friday, Saturday, and Sunday will be exactly 80°F, or else each guest pays only half price for the facilities. If the temperatures on Friday and Saturday were 90°F and 82°F, respectively, what must the temperature be on Sunday so that the resort does not lose half of its revenue?
10. A patient's temperature was taken at 6 A.M., 12 noon, 3 P.M., and 8 P.M. The first, third, and fourth readings were 102.5°, 101.5°, and

102°F, respectively. The nurse forgot to write down the second reading, but recorded that the average of the four readings was 101.5°F. What was the second temperature reading?

11. A 12-meter long steel beam is to be cut into two pieces so that one piece will be 4 meters longer than the other. How long will each piece be?

12. A rectangular field whose length is 10 meters longer than its width is to be enclosed with exactly 100 meters of fencing material. What are the dimensions of the field?

13. A vending machine contains $3.00 in nickels and dimes. If the number of dimes is 5 more than twice the number of nickels, how many coins of each type are there?

14. A wallet contains $460 in $5, $10, and $20 bills. The number of $5 bills exceeds twice the number of $10 bills by 4, and the number of $20 bills is 6 fewer than the number of $10 bills. How many bills of each type are there?

15. A movie theater charges $7.50 admission for an adult and $5 for a child. If 700 tickets are sold on a particular day and the total revenue received is $4500, how many tickets of each type are sold?

16. A student bought 23-cent, 41-cent, and 80-cent stamps with a total value of $31.50. If the number of 23-cent stamps is 2 more than the number of 41-cent stamps, and the number of 80-cent stamps is 5 more than one-half the number of 41-cent stamps, how many stamps of each denomination did the student obtain?

17. An amateur theater group is converting a classroom to an auditorium for a forthcoming play. The group sells $3, $5, and $6 tickets, and receives exactly $503 from the sale of tickets. If the number of $5 tickets is twice the number of $6 tickets, and the number of $3 tickets is 1 more than 3 times the number of $6 tickets, how many tickets of each type are there?

18. To pay for their child's college education, the parents invested $10,000, part in a certificate of deposit paying 8.5% annual interest, the rest in a mutual fund paying 7% annual interest. The annual income from the certificate of deposit is $200 more than the annual income from the mutual fund. How much money was put into each type of investment?

19. A bicycle store is closing out its entire stock of a certain brand of three-speed and ten-speed models. The profit on a three-speed bicycle is 11% of the sale price, and the profit on a ten-speed model is 22% of the sale price. If the entire stock is sold for $16,000 and the profit on the entire stock is 19%, how much is obtained from the sale of each type of bicycle?

20. A film shop carrying black-and-white film and color film has $4000 in inventory. The profit on black-and-white film is 12%, and the profit on color film is 21%. If all the film is sold, and if the profit on color film is $150 less than the profit on black-and-white film, how much was invested in each type of film?

21. A firm borrowed $12,000 at a simple annual interest rate of 8% for a period of 3 years. At the end of the first year, the firm found that its needs were reduced. The firm returned a portion of the original loan and retained the remainder until the end of the 3-year period. If the total interest paid was $1760, how much was returned at the end of the first year?

22. A finance company lent a certain amount of money to Firm A at 7% annual interest. An amount $100 less than that lent to Firm A was lent to Firm B at 8%, and an amount $200 more than that lent to Firm A was lent to Firm C at 8.5%. All loans were for one year. If the total annual income is $126.50, how much was lent to each firm?

23. Two trucks leave Philadelphia for Miami. The first truck to leave travels at an average speed of 50 kilometers per hour. The second truck, which leaves 2 hours later, travels at an average speed of 55 kilometers per hour.

How long does it take the second truck to overtake the first truck?

24. Jackie either drives or bicycles from home to school. Her average speed when driving is 36 mph, and her average speed when bicycling is 12 mph. If it takes her $\frac{1}{2}$ hour less to drive to school than to bicycle, how long does it take her to go to school, and how far is the school from her home?

25. Professors Roberts and Jones, who live 676 miles apart, are exchanging houses and jobs for the summer. They start out for their new locations at exactly the same time, and they meet after 6.5 hours of driving. If their average speeds differ by 4 mph, what are their average speeds?

26. Steve leaves school by moped for spring vacation. Forty minutes later his roommate, Frank, notices that Steve forgot to take his camera. So, Frank decides to try to catch up with Steve by car. If Steve's average speed is 25 mph and Frank averages 45 mph, how long does it take Frank to overtake Steve?

27. An express train and a local train start out from the same point at the same time and travel in opposite directions. The express train travels twice as fast as the local train. If after 4 hours they are 480 kilometers apart, what is the average speed of each train?

28. How many pounds of raisins worth \$3 per pound must be mixed with 10 pounds of peanuts worth \$2.40 per pound to produce a mixture worth \$2.80 per pound?

29. How many ounces of Ceylon tea worth \$1.50 per ounce and how many ounces of Formosa tea worth \$2.00 per ounce must be mixed to obtain a mixture of 8 ounces that is worth \$1.85 per ounce?

30. A copper alloy that is 40% copper is to be combined with a copper alloy that is 80% copper to produce 120 kilograms of an alloy that is 70% copper. How many kilograms of each alloy must be used?

31. A vat contains 27 gallons of water and 9 gallons of acetic acid. How many gallons of water must be evaporated if the resulting solution is to be 40% acetic acid?

32. A producer of packaged frozen vegetables wants to market mixed vegetables at \$1.20 per kilogram. How many kilograms of green beans worth \$1.00 per kilogram must be mixed with 100 kilograms of corn worth \$1.30 per kilogram and 90 kilograms of peas worth \$1.40 per kilogram to produce a satisfactory mixture?

33. A certain number is 3 times another. If the difference of their reciprocals is 8, find both numbers.

34. If $\frac{1}{3}$ is subtracted from 3 times the reciprocal of a certain number, the result is $\frac{25}{6}$. Find the number.

35. Computer A can carry out an engineering analysis in 6 hours, but computer B can do the same job in 4 hours. How long does it take to complete the job if both computers work together?

36. Jackie can paint a certain room in 3 hours, Lisa in 4 hours, and Susan in 2 hours. How long does it take to paint the room if they all work together?

37. A senior copy editor together with a junior copy editor can edit a book in 3 days. The junior editor, working alone, would take twice as long to complete the job as the senior editor would require if working alone. How long would it take each editor to complete the job by herself?

38. Hose A can fill a certain vat in 3 hours. After 2 hours of pumping, hose A is turned off. Hose B is then turned on and completes filling the vat in 3 more hours. How long would it take hose B to fill the vat alone?

39. A printing shop starts a job at 10 A.M. on press A. Using this press alone, it would take 8 hours to complete the job. At 2 P.M. press B is also turned on, and both presses together

finish the job at 4 P.M. How long would it take press B to do the job alone?

40. A boat travels 20 kilometers upstream in the same time that it would take the same boat to travel 30 kilometers downstream. If the rate of the stream is 5 kilometers per hour, find the speed of the boat in still water.

41. An airplane flying against the wind travels 300 miles in the same time that it would take the same plane, flying the same speed, to travel 400 miles with the wind. If the wind speed is 20 mph, find the speed of the airplane in still air.

In Exercises 42–51 solve for the indicated variable in terms of the remaining variables.

42. $A = Pr$ for r

43. $C = 2\pi r$ for r

44. $V = \frac{1}{3}\pi r^2 h$ for h

45. $F = \frac{9}{5}C + 32$ for C

46. $S = \frac{1}{2}gt^2 + vt$ for v

47. $A = \frac{1}{2}h(b + b')$ for b

48. $A = P(1 + rt)$ for r

49. $\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$ for f_2

50. $a = \frac{v_1 - v_0}{t}$ for v_0

51. $S = \frac{a - rL}{L - r}$ for L

52. Translate the following from words to an algebraic expression or equation, denoting the unknown by n.
   a. The express train travels 5 mph faster than the local train.
   b. The length of a rectangle is 7 inches more than its width.
   c. the area of a triangle, if the altitude is twice the base
   d. the sum of 3 consecutive even numbers
   e. 15% of the amount by which a number exceeds 10,000

53. If r and s represent two numbers, write the following:
   a. twice the sum of the two numbers
   b. 5% of the difference between the two numbers
   c. 5 less than twice the second number
   d. the ratio of the first to the second number
   e. the sum of the squares of the two numbers
   f. the average of the two numbers
   g. 6 times the first number less 4 times the second number

54. Write formulas for each of the following:
   a. the charge in cents for a telephone call between two cities lasting n minutes, n greater than 3, if the charge for the first 3 minutes is \$1.20 and each additional minute costs 33 cents
   b. the taxi fare for m miles, if the initial charge is \$2.50 and the driver charges 70 cents for every $\frac{1}{5}$ mile traveled
   c. the amount in an account at the end of a year, if simple interest is paid at the rate of 16%, and the account contains d dollars at the beginning of the year
   d. the fine a company paid for dumping acid into the Mississippi River for d days, if the U.S. Environmental Protection Agency fined the company \$150,000 plus \$1000 per day until the company complied with the federal water pollution regulations.

55. Find three consecutive even numbers such that twice the first plus 3 times the second is 4 times the third.

56. When exercising, Mary walks a distance to warm up, jogs $3\frac{1}{2}$ times as far as she walks, and sprints $3\frac{1}{3}$ times as far as she jogs. If she covers 4171 meters, find the distances that she walked, jogged, and sprinted.

57. A 10-quart radiator has 30% antifreeze. How much of the fluid should be drained and replaced with pure antifreeze to double the strength of the mixture?

58. There are two identical beakers in a chemistry laboratory, both filled to the same level. One contains sulfuric acid and the other contains water. First, one spoon of acid is put into the beaker with the water and mixed thoroughly. Then one spoon of this mixture is put back into the beaker with the acid. Is there more water in the acid or more acid in the water?

59. Two bicyclists leave cities A and B at the same time, heading toward each other. Their speeds are 20 mph and 30 mph, respectively. The distance between these cities is 100 miles. Simultaneously, a bird leaves city A, heading toward B, traveling at 40 mph. When it meets the bicyclist who left from B, it turns around and heads back toward A. When it subsequently meets the bicyclist who came from A, it turns around and heads back toward B, and so on. Find the total distance the bird will have flown by the time the two bicyclists meet.

60. To determine the number of deer in a forest, a conservationist catches 225 deer, tags them and then releases them. A week later, 102 deer are caught and, of those, 15 are found to be tagged. Assuming that the proportion of tagged deer in the second sample was the same as the proportion of all tagged deer in the total population, estimate the number of deer in the forest.

61. In a Tour de France bicycle race, Stefan averaged 20 mph for the first third of the race and 35 mph for the remainder. Enrique maintained a constant speed of 30 mph throughout the race. Of these two, who finished first?

In Exercises 62–65, solve for the indicated variable.

62. $I = \dfrac{E}{r + \dfrac{R}{n}}$ for n

63. $Wf = \left(\dfrac{W}{k} - 1\right)\left(\dfrac{1}{k}\right)$ for W

64. $W = \dfrac{2PR}{R - r}$ for r

65. $\dfrac{E}{c} = \dfrac{R + r}{r}$ for r

2.3 The Quadratic Equation

We now turn our attention to equations involving second-degree polynomials. A **quadratic equation** is an equation of the form

$$ax^2 + bx + c = 0, \qquad a \neq 0$$

where a, b, and c are real numbers. In this section we will explore techniques for solving this important class of equations. We will also show that there are several kinds of equations that can be transformed into quadratic equations and then solved.

Solving by Factoring

If we can factor the left-hand side of the quadratic equation

$$ax^2 + bx + c = 0, \qquad a \neq 0$$

into two linear factors, then we can solve the equation. For example, the quadratic equation

$$x^2 - 5x + 6 = 0$$

can be written as

$$(x - 2)(x - 3) = 0$$

since 0 is the only number with the following property:

If $ab = 0$, then $a = 0$ or $b = 0$.

We can set each factor of the above quadratic equation equal to 0.

$$x - 2 = 0 \quad \text{or} \quad x - 3 = 0$$
$$x = 2 \quad \text{or} \quad x = 3$$

The solutions of the given quadratic equation are 2 and 3.

EXAMPLE 1 SOLVING BY FACTORING

Solve the equation $2x^2 - 3x - 2 = 0$ by factoring.

SOLUTION

Factoring, we have

$$2x^2 - 3x - 2 = 0$$
$$(2x + 1)(x - 2) = 0$$

Since the product of the factors is 0, at least one factor must be 0. Setting each factor equal to 0, we have

$$2x + 1 = 0 \quad \text{or} \quad x - 2 = 0$$
$$x = -\frac{1}{2} \quad \text{or} \quad x = 2$$

EXAMPLE 2 SOLVING BY FACTORING

Solve the equation $3x^2 + 5x - 2 = 0$ by factoring.

SOLUTION

Factoring, we have

$$(3x - 1)(x + 2) = 0$$
$$3x - 1 = 0 \quad \text{or} \quad x + 2 = 0$$
$$x = \frac{1}{3} \quad \text{or} \quad x = -2$$

EXAMPLE 3 SOLVING BY FACTORING

Solve the equation $3x^2 - 4x = 0$ by factoring.

SOLUTION

Factoring, we have

$$3x^2 - 4x = 0$$
$$x(3x - 4) = 0$$

Setting each factor equal to zero,

$$x = 0 \qquad \text{or} \qquad x = \frac{4}{3}$$

WARNING

When considering an equation with a common factor, such as

$$3x^2 - 4x = 0$$
$$x(3x - 4) = 0$$

always set each factor equal to zero. A common error of students is to divide both sides of the above equation by x.

$$\frac{x(3x - 4)}{x} = \frac{0}{x}$$

concluding that

$$3x - 4 = 0$$
$$x = \frac{4}{3}$$

The only time this operation is permitted is if $x \neq 0$. If $x = 0$ were possible, then you would have "lost" this root of the original equation.

✔ Progress Check

Solve each of the given equations by factoring.

a. $4x^2 - x = 0$ b. $3x^2 - 11x - 4 = 0$

Answers

a. $0, \frac{1}{4}$ b. $-\frac{1}{3}, 4$

One cannot always find "simple" factors to solve quadratic equations. For the most part, we will only attempt to use the factoring method for general quadratic equations with rational roots. In those cases, the factors only have integer coefficients.

Furthermore, there are some quadratic equations that cannot even be factored over the real numbers. Consider using the factoring method to solve

$$x^2 + x + 1 = 0$$

There do not exist any real numbers r and s such that

$$x^2 + x + 1 = (x + r)(x + s)$$

However, it can be written in this form if we permit r and s to be complex numbers. For this reason, it is necessary to develop solution techniques that are more powerful than factoring.

Special Cases: $x^2 - p = 0$, $x^2 + p = 0$, $a(x+h)^2 + c = 0$

There are certain quadratic equations that do not necessarily require the use of factoring when finding solutions. Because of their special form, we may use the method of taking roots.

EXAMPLE 4 SPECIAL CASES

Solve the equation $x^2 - 3 = 0$.

SOLUTION

We may write the original equation as

$$x^2 = 3$$

Taking the square root of both sides, we obtain

$$x = \sqrt{3} \qquad \text{or} \qquad x = -\sqrt{3}$$

Sometimes these solutions are written in the abbreviated form $x = \pm\sqrt{3}$.

Alternatively, if we recognize that the original equation can be factored as

$$x^2 - 3 = (x - \sqrt{3})(x + \sqrt{3}) = 0$$

we obtain the same results by setting each factor equal to 0.

For a positive number p, consider $x^2 - p = 0$ or, equivalently, $x^2 = p$. Taking the square root of both sides of this equation, we obtain $x = \pm\sqrt{p}$. Alternatively, we may factor

$$x^2 - p = (x - \sqrt{p})(x + \sqrt{p})$$

(Check this by multiplying the factors of the right-hand side of the equation.) This leads to the following result.

If $p > 0$ and $x^2 = p$, then $x = \pm\sqrt{p}$. Furthermore, the equation can be written in factored form as

$$x^2 - p = (x - \sqrt{p})(x + \sqrt{p}) = 0$$

EXAMPLE 5 SPECIAL CASES

Solve the equation $x^2 + 4 = 0$.

SOLUTION

We may write the original equation as

$$x^2 = -4$$

Taking the square root of both sides, we obtain

$$x = \pm 2i$$

Alternatively, if we recognize that the original equation can be factored as

$$x^2 + 4 = (x - 2i)(x + 2i) = 0$$

we obtain the same results by setting each factor equal to 0.

For a positive number p, consider $x^2 + p = 0$ or, equivalently, $x^2 = -p$. Taking the square root of both sides of this equation we obtain $x = \pm\sqrt{p}i$. Alternatively, we may factor

$$x^2 + p = (x - \sqrt{p}i)(x + \sqrt{p}i)$$

(Check by multiplying the factors of the right-hand side.)

This leads to the following result.

If $p > 0$ and $x^2 = -p$, then $x = \pm\sqrt{p}i$. Furthermore, the equation can be written in factored form as

$$x^2 + p = (x - \sqrt{p}i)(x + \sqrt{p}i) = 0$$

EXAMPLE 6 SPECIAL CASES

Solve the equation $2x^2 - 6 = 0$.

SOLUTION

$$
\begin{aligned}
2x^2 - 6 &= 0 \\
2x^2 &= 6 \\
x^2 &= 3 \\
x &= \pm\sqrt{3}
\end{aligned}
$$

EXAMPLE 7 SPECIAL CASES

Solve the equation $2(x - 1)^2 - 6 = 0$.

SOLUTION

$$
\begin{aligned}
2(x - 1)^2 - 6 &= 0 \\
2(x - 1)^2 &= 6 \\
(x - 1)^2 &= 3 \\
x - 1 &= \pm\sqrt{3} \\
x &= 1 \pm \sqrt{3}
\end{aligned}
$$

EXAMPLE 8 SPECIAL CASES

Solve the equation $4(x + 3)^2 + 20 = 0$.

SOLUTION

$$
\begin{aligned}
4(x + 3)^2 + 20 &= 0 \\
4(x + 3)^2 &= -20 \\
(x + 3)^2 &= -5 \\
x + 3 &= \pm\sqrt{5}\,i \\
x &= -3 \pm \sqrt{5}\,i
\end{aligned}
$$

Equations of the form $a(x + h)^2 + c = 0$ may be solved using the techniques of Examples 7 and 8 as follows:

$$
\begin{aligned}
a(x + h)^2 + c &= 0 \\
a(x + h)^2 &= -c \\
(x + h)^2 &= -\frac{c}{a} \\
x + h &= \pm\sqrt{-\frac{c}{a}} \\
x &= -h \pm \sqrt{-\frac{c}{a}}
\end{aligned}
$$

✔ Progress Check

Solve the given equation.

a. $5x^2 + 13 = 0$ b. $(2x - 7)^2 - 5 = 0$

Answers

a. $\pm \dfrac{\sqrt{65}}{5} i$ b. $\dfrac{7 \pm \sqrt{5}}{2}$

We have seen that the solutions of a quadratic equation may be complex numbers, whereas the solution of a linear equation is a real number. In addition, quadratic equations appear to have two solutions. We will have more to say about these observations when we study the roots of polynomial equations in a later chapter.

We have shown that we can always find a solution to a quadratic equation of the form

$$a(x + h)^2 + c = 0 \tag{1}$$

A technique known as **completing the square** permits us to rewrite *any* quadratic equation in the form of Equation (1). Beginning with the expression $x^2 + dx$, we seek a constant h^2 to complete the square so that

$$x^2 + dx + h^2 = (x + h)^2$$

Expanding and solving, we have

$$\begin{aligned} x^2 + dx + h^2 &= x^2 + 2hx + h^2 \\ dx &= 2hx \\ h &= \frac{d}{2} \\ h^2 &= \left(\frac{d}{2}\right)^2 \end{aligned}$$

so $h^2 = \left(\dfrac{d}{2}\right)^2$ is the amount to be added to $x^2 + dx$ to form a perfect square.

EXAMPLE 9 COMPLETING THE SQUARE

Complete the square for each of the following.

a. $x^2 - 6x$ b. $x^2 + 3x$

SOLUTION

a. The coefficient of x is -6, so $h = -\dfrac{6}{2} = -3$ and $h^2 = 9$. Then

$$x^2 - 6x + 9 = (x - 3)^2$$

b. The coefficient of x is 3, and $h^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}$. Then

$$x^2 + 3x + \frac{9}{4} = \left(x + \frac{3}{2}\right)^2.$$

We are now in a position to use this method to solve a quadratic equation.

EXAMPLE 10 COMPLETING THE SQUARE

Solve the quadratic equation $2x^2 - 10x + 1 = 0$ by completing the square.

SOLUTION

We outline and explain each step of the process in Table 2.

TABLE 2 Completing the Square

Method	Example
Step 1. Rewrite the equation with the constant term on the right-hand side.	*Step 1.* $2x^2 - 10x = -1$
Step 2. Factor out a, the coefficient of x^2.	*Step 2.* $2(x^2 - 5x) = -1$
Step 3. Divide both sides of the equation by a.	*Step 3.* $x^2 - 5x = -\frac{1}{2}$
Step 4. Find $h = \frac{d}{2}$ and $h^2 = (\frac{d}{2})^2$, where d is the coefficient of x in *Step 3*.	*Step 4.* $h = \frac{-5}{2}, \quad h^2 = \frac{25}{4}$
Step 5. Add h^2 to both sides of the equation.	*Step 5.* $x^2 - 5x + \frac{25}{4} = -\frac{1}{2} + \frac{25}{4}$
Step 6. Simplify.	*Step 6.* $\left(x - \frac{5}{2}\right)^2 = \frac{23}{4}$
Step 7. Solve for x.	*Step 7.* $x - \frac{5}{2} = \pm\sqrt{\frac{23}{4}}$ $x = \frac{5}{2} \pm \frac{\sqrt{23}}{2}$ $x = \frac{5 \pm \sqrt{23}}{2}$

✔ Progress Check

Solve by completing the square.

a. $x^2 - 3x + 2 = 0$ b. $3x^2 - 4x + 2 = 0$

Answers

a. 1, 2 b. $\frac{2 \pm \sqrt{2}i}{3}$

The Quadratic Formula

We can apply the method of completing the square to the general quadratic equation

$$ax^2 + bx + c = 0, \qquad a \neq 0$$

Following the steps of the method as shown in Table 2, we proceed as follows:

1. Move the constant term to the right-hand side. $\quad ax^2 + bx = -c$
2. Factor out a, the coefficient of x^2. $\quad a\left(x^2 + \frac{b}{a}x\right) = -c$
3. Divide both sides of the equation by a. $\quad x^2 + \frac{b}{a}x = -\frac{c}{a}$
4. Find $h = \frac{d}{2}$ and $h^2 = (\frac{d}{2})^2$, where d is the coefficient of x. $\quad h = \frac{b}{2a}, \quad h^2 = \frac{b^2}{4a^2}$
5. Add h^2 to both sides of the equation. $\quad x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$
6. Simplify.

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c(4a)}{a(4a)}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

7. Solve for x.

$$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

$$x = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

The quadratic equation

$$ax^2 + bx + c = 0, \qquad a \neq 0$$

has *two* roots often written in the compact form

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \qquad a \neq 0$$

EXAMPLE 11 THE QUADRATIC FORMULA

Solve $2x^2 - 3x - 3 = 0$ by the quadratic formula.

SOLUTION

Since $a = 2$, $b = -3$ and $c = -3$, we have

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-3)}}{2(2)}$$

$$x = \frac{3 \pm \sqrt{33}}{4}$$

EXAMPLE 12 THE QUADRATIC FORMULA

Solve $-5x^2 + 3x = 2$ by the quadratic formula.

SOLUTION

We first rewrite the given equation as $-5x^2 + 3x - 2 = 0$. Then $a = -5$, $b = 3$ and $c = -2$. Substituting into the quadratic formula, we have

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{3^2 - 4(-5)(-2)}}{2(-5)}$$

$$x = \frac{-3 \pm \sqrt{-31}}{-10}$$

$$x = \frac{-3 \pm \sqrt{31}\, i}{-10}$$

(Show that this is equivalent to $x = \dfrac{3 \pm \sqrt{31}\, i}{10}$.)

✔ Progress Check

Solve by the quadratic formula.

a. $x^2 - 8x = -10$ b. $4x^2 - 2x + 1 = 0$

Answers

a. $4 \pm \sqrt{6}$ b. $\dfrac{1 \pm \sqrt{3}i}{4}$

WARNING

There are a number of errors that students make in using the quadratic formula.

a. To solve $x^2 - 3x = -4$, you must write the equation in the form

$$x^2 - 3x + 4 = 0$$

to properly identify a, b, and c. Note that $b = -3$, *not* 3.

b. The quadratic formula is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note that

$$x \neq -b \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

since the term $-b$ must also be divided by $2a$.

Now that there is a formula that works for any quadratic equation, it may be tempting to use it all the time. However, if you see an equation such as

$$x^2 = 15$$

it may be easier to obtain the answer: $x = \pm\sqrt{15}$. Similarly, when faced with

$$x^2 + 3x + 2 = 0$$

it may be faster to solve it if you see that

$$x^2 + 3x + 2 = (x + 1)(x + 2)$$

The method of completing the square is generally not used for solving quadratic equations once the quadratic formula is learned. The *technique* of completing the square is helpful in a variety of applications, and we will use it in a later chapter when we graph second-degree equations.

The Discriminant

By analyzing the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

we can learn a great deal about the roots of the quadratic equation

$$ax^2 + bx + c = 0, \qquad a \neq 0$$

The key to the analysis is the **discriminant** $b^2 - 4ac$ found under the radical.

- If $b^2 - 4ac$ is negative, we have the square root of a negative number, and the roots of the quadratic equation are complex numbers as conjugate pairs.

- If $b^2 - 4ac$ is positive, we have the square root of a positive number, and the roots of the quadratic equation are two different real numbers.
- If $b^2 - 4ac = 0$, then $x = -\dfrac{b}{2a}$, which we call a **double root** or **repeated root** of the quadratic equation. For example, if $x^2 - 10x + 25 = 0$, then the discriminant is 0 and $x = 5$. However,

$$x^2 - 10x + 25 = (x - 5)(x - 5) = 0$$

We call $x = 5$ a double root because the factor $x - 5$ is a double factor of $x^2 - 10x + 25 = 0$. This hints at the importance of the relationship between roots and factors, a relationship that we will explore later in Chapter 4. We summarize in Table 3.

TABLE 3 Discriminant-Root Analysis

The quadratic equation $ax^2 + bx + c = 0,\ a \neq 0$, has exactly two roots, the nature of which are determined by the discriminant $b^2 - 4ac$.

Discriminant	Roots
Negative	Two complex roots as conjugate pairs
0	One real double root
Positive	Two different real roots

If the roots of the quadratic equation are real, and a, b, and c are rational numbers, the discriminant enables us to determine whether the roots are rational or irrational. Since $\sqrt{k}$ is a rational number only if k is a perfect square, we see that the quadratic formula produces a rational result only if $b^2 - 4ac$ is a perfect square.

EXAMPLE 13 DISCRIMINANT-ROOT ANALYSIS

Without solving, determine the nature of the roots of the quadratic equation $3x^2 - 4x + 6 = 0$.

SOLUTION

We evaluate $b^2 - 4ac$ using $a = 3$, $b = -4$ and $c = 6$. Thus,

$$b^2 - 4ac = (-4)^2 - 4(3)(6) = 16 - 72 = -56$$

The discriminant is negative, so the equation has two complex roots.

EXAMPLE 14 DISCRIMINANT-ROOT ANALYSIS

Without solving, determine the nature of the roots of the equation $2x^2 - 7x = -1$.

SOLUTION

We rewrite the equation in the standard form

$$2x^2 - 7x + 1 = 0$$

and then substitute $a = 2$, $b = -7$ and $c = 1$ in the discriminant. Therefore,

$$b^2 - 4ac = (-7)^2 - 4(2)(1) = 49 - 8 = 41$$

The discriminant is positive and is not a perfect square. Thus, the roots are real, unequal, and irrational.

✔ Progress Check

Without solving, determine the nature of the roots of the quadratic equation by using the discriminant.

a. $4x^2 - 20x + 25 = 0$

b. $5x^2 - 6x = -2$

c. $10x^2 = x + 2$

d. $x^2 + x - 1 = 0$

Answers

a. a real, double root

b. two complex roots

c. two real, rational roots

d. two real, irrational roots

Forms Leading to Quadratics

Certain types of equations can be transformed into quadratic equations that can be solved by the methods discussed in this section. One form that leads to a quadratic equation is the **radical equation,** such as

$$x - \sqrt{x - 2} = 4$$

which is solved in Example 15. To solve the equation, we isolate the radical and raise both sides to a suitable power. The following is the key to the solution of such equations:

If P and Q are algebraic expressions, then the solution set of the equation

$$P = Q$$

is a subset of the solution set of the equation

$$P^n = Q^n$$

where n is a natural number.

This suggests that we can solve radical equations if we observe a precaution.

> If both sides of an equation are raised to the same power, the solutions of the resulting equation must be checked to see that they satisfy the original equation.

EXAMPLE 15 RADICAL EQUATIONS

Solve $x - \sqrt{x-2} = 4$.

SOLUTION

Solving Radical Equations

Step 1. If possible, isolate the radical on one side of the equation.

Step 1.
$$x - 4 = \sqrt{x-2}$$

Step 2. Raise both sides of the equation to a suitable power to eliminate the radical. If necessary, go back to *Step 1*.

Step 2. Squaring both sides, we have
$$x^2 - 8x + 16 = x - 2$$

Step 3. Solve for the unknown.

Step 3.
$$x^2 - 9x + 18 = 0$$
$$(x-3)(x-6) = 0$$
$$x = 3 \qquad x = 6$$

Step 4. Check each solution by substituting in the *original* equation.

Step 4. $\text{LHS} = x - \sqrt{x-2} \qquad \text{RHS} = 4$

Check $x = 3$
$$\text{LHS} = 3 - \sqrt{3-2} = 2 \neq 4 = \text{RHS}$$
Check $x = 6$
$$\text{LHS} = 6 - \sqrt{6-2} = 6 - 2 = 4 = \text{RHS}$$

We will use the abbreviation LHS for left-hand side and RHS for right-hand side. We conclude that 6 is a solution of the original equation, and 3 is not a solution of the original equation. We say that 3 is an **extraneous solution** that was introduced when we raised both sides of the original equation to the second power.

✔ Progress Check

Solve $x - \sqrt{1-x} = -5$.

Answer

-3

The equation in the next example contains more than one radical. Solving this equation requires that we square both sides *twice*.

EXAMPLE 16 RADICAL EQUATIONS

Solve $\sqrt{2x-4} - \sqrt{3x+4} = -2$.

SOLUTION

Before squaring, rewrite the equation so that we isolate one of the radicals on one side of the equation.

$$\begin{aligned}
\sqrt{2x-4} &= \sqrt{3x+4} - 2 & \\
2x - 4 &= (3x+4) - 4\sqrt{3x+4} + 4 & \text{Square both sides.}\\
-x - 12 &= -4\sqrt{3x+4} & \text{Isolate the radical.}\\
x^2 + 24x + 144 &= 16(3x+4) & \text{Square both sides.}\\
x^2 - 24x + 80 &= 0 & \\
(x-20)(x-4) &= 0 & \\
x = 20 \qquad x &= 4 &
\end{aligned}$$

Verify that both 20 and 4 are solutions of the original equation.

✔ Progress Check

Solve $\sqrt{5x-1} - \sqrt{x+2} = 1$.

Answer

2

Although the equation

$$x^4 - x^2 - 2 = 0$$

is not a quadratic in the unknown x, it is a quadratic in the unknown x^2:

$$(x^2)^2 - (x^2) - 2 = 0$$

This may be seen more clearly by replacing x^2 with a new unknown u such that $u = x^2$. Substituting, we have

$$u^2 - u - 2 = 0$$

which is a quadratic equation in the unknown u. Solving, we find

$$(u+1)(u-2)=0$$
$$u=-1 \quad \text{or} \quad u=2$$

Since $x^2 = u$, we must next solve the equations

$$x^2=-1 \qquad \text{and} \qquad x^2=2$$
$$x=\pm i \qquad\qquad x=\pm\sqrt{2}$$

The original equation has four solutions: i, $-i$, $\sqrt{2}$, and $-\sqrt{2}$.

The technique we have used is called a **substitution of variable**. This is a powerful method that is commonly used in calculus.

✔ Progress Check

Indicate an appropriate substitution of variable and solve each of the following equations:

a. $3x^4 - 10x^2 - 8 = 0$

b. $4x^{2/3} + 7x^{1/3} - 2 = 0$

c. $\dfrac{2}{x^2} + \dfrac{1}{x} - 10 = 0$

d. $\left(1+\dfrac{2}{x}\right)^2 - 8\left(1+\dfrac{2}{x}\right) + 15 = 0$

Answers

a. $u = x^2$; ± 2, $\pm\dfrac{\sqrt{6}i}{3}$

b. $u = x^{1/3}$; $\dfrac{1}{64}$, -8

c. $u = \dfrac{1}{x}$; $-\dfrac{2}{5}$, $\dfrac{1}{2}$

d. $u = 1 + \dfrac{2}{x}$; 1, $\dfrac{1}{2}$

Graphing Calculator Alert

There are programs available for most graphing calculators which allow the user to input the values of a, b, and c, and then display the solutions. Look online for a program you could download. You may find a site which gives you the lines of code needed; you would then write the program line-by-line using the “edit” option in your graphing calculator’s program menu.

Exercise Set 2.3

In Exercises 1–14, solve by factoring.

1. $x^2 - 3x + 2 = 0$
2. $x^2 - 6x + 8 = 0$
3. $x^2 + x - 2 = 0$
4. $3r^2 - 4r + 1 = 0$
5. $x^2 + 6x = -8$
6. $x^2 + 6x + 5 = 0$

7. $y^2 - 4y = 0$
8. $2x^2 - x = 0$
9. $2x^2 - 5x = -2$
10. $2s^2 - 5s - 3 = 0$
11. $t^2 - 4 = 0$
12. $4x^2 - 9 = 0$
13. $6x^2 - 5x + 1 = 0$
14. $6x^2 - x = 2$

In Exercises 15–24, solve the given equation.

15. $3x^2 - 27 = 0$
16. $4x^2 - 64 = 0$
17. $5y^2 - 25 = 0$
18. $6x^2 - 12 = 0$
19. $(2r + 5)^2 = 8$
20. $(3x - 4)^2 = -6$
21. $(3x - 5)^2 - 8 = 0$
22. $(4t + 1)^2 - 3 = 0$
23. $9x^2 + 64 = 0$
24. $81x^2 + 25 = 0$

In Exercises 25–36, solve by completing the square.

25. $x^2 - 2x = 8$
26. $t^2 - 2t = 15$
27. $2r^2 - 7r = 4$
28. $9x^2 + 3x = 2$
29. $3x^2 + 8x = 3$
30. $2y^2 + 4y = 5$
31. $2y^2 + 2y = -1$
32. $3x^2 - 4x = -3$
33. $4x^2 - x = 3$
34. $2x^2 + x = 2$
35. $3x^2 + 2x = -1$
36. $3u^2 - 3u = -1$

In Exercises 37–48, solve by the quadratic formula.

37. $2x^2 + 3x = 0$
38. $2x^2 + 3x + 3 = 0$
39. $5x^2 - 4x + 3 = 0$
40. $2x^2 - 3x - 2 = 0$
41. $5y^2 - 4y + 5 = 0$
42. $x^2 - 5x = 0$
43. $3x^2 + x - 2 = 0$
44. $2x^2 + 4x - 3 = 0$
45. $3y^2 - 4 = 0$
46. $2x^2 + 2x + 5 = 0$
47. $4u^2 + 3u = 0$
48. $4x^2 - 1 = 0$

In Exercises 49–58, solve by any method.

49. $2x^2 + 2x - 5 = 0$
50. $2t^2 + 2t + 3 = 0$
51. $3x^2 + 4x - 4 = 0$
52. $x^2 + 2x = 0$
53. $2x^2 + 5x + 4 = 0$
54. $2r^2 - 3r + 2 = 0$
55. $4u^2 - 1 = 0$
56. $x^2 + 2 = 0$
57. $4x^3 + 2x^2 + 3x = 0$
58. $4s^3 + 4s^2 - 15s = 0$

In Exercises 59–64, solve for the indicated variable in terms of the remaining variables.

59. $a^2 + b^2 = c^2$, for b
60. $s = \frac{1}{2}gt^2$, for t
61. $V = \frac{1}{3}\pi r^2 h$, for r
62. $A = \pi r^2$, for r
63. $s = \frac{1}{2}gt^2 + vt$, for t
64. $F = g\dfrac{m_1 m_2}{d^2}$, for d

Without solving, determine the nature of the roots of each quadratic equation in Exercises 65–80.

65. $x^2 - 2x + 3 = 0$
66. $3x^2 + 2x - 5 = 0$
67. $4x^2 - 12x + 9 = 0$
68. $2x^2 + x + 5 = 0$
69. $-3x^2 + 2x + 5 = 0$
70. $-3y^2 + 2y - 5 = 0$
71. $3x^2 + 2x = 0$
72. $4x^2 + 20x + 25 = 0$
73. $2r^2 = r - 4$
74. $3x^2 = 5 - x$
75. $3x^2 + 6 = 0$
76. $4x^2 - 25 = 0$
77. $6r = 3r^2 + 1$
78. $4x = 2x^2 + 3$
79. $12x = 9x^2 + 4$
80. $4s^2 = -4s - 1$

In Exercises 81–84, find a value or values of k for which the quadratic has a double root.

81. $kx^2 - 4x + 1 = 0$
82. $2x^2 + 3x + k = 0$
83. $x^2 - kx - 2k = 0$
84. $kx^2 - 4x + k = 0$

In Exercises 85–92, find the solution set.

85. $x + \sqrt{x + 5} = 7$
86. $x - \sqrt{13 - x} = 1$
87. $2x + \sqrt{x + 1} = 8$
88. $3x - \sqrt{1 + 3x} = 1$
89. $\sqrt{3x + 4} - \sqrt{2x + 1} = 1$
90. $\sqrt{4 - 4x} - \sqrt{x + 4} = 3$

91. $\sqrt{2x-1}+\sqrt{x-4}=4$

92. $\sqrt{5x+1}+\sqrt{4x-3}=7$

In Exercises 93–100, indicate an appropriate substitution of variable and solve each of the equations.

93. $3x^4+5x^2-2=0$

94. $2x^6+15x^3-8=0$

95. $\dfrac{6}{x^2}+\dfrac{1}{x}-2=0$

96. $\dfrac{2}{x^4}-\dfrac{3}{x^2}-9=0$

97. $2x^{2/5}+5x^{1/5}+2=0$

98. $3x^{4/3}-4x^{2/3}-4=0$

99. $2\left(\dfrac{1}{x}+1\right)^2-3\left(\dfrac{1}{x}+1\right)-20=0$

100. $3\left(\dfrac{1}{x}-2\right)^2+2\left(\dfrac{1}{x}-2\right)-1=0$

In Exercises 101 and 102, provide a proof of the following statements.

101. If r_1 and r_2 are the roots of the equation $ax^2+bx+c=0$, then (a) $r_1r_2=\frac{c}{a}$ and (b) $r_1+r_2=\frac{b}{a}$.

102. If a, b, and c are rational numbers, and the discriminant of the equation $ax^2+bx+c=0$ is positive, then the quadratic has either two rational roots or two irrational roots.

In Exercises 103–109, use the theorems of Exercise 101 to find a value or values of k that satisfies the indicated condition.

103. $kx^2+3x+5=0$; sum of the roots is 6.

104. $2x^2-3kx-2=0$; sum of the roots is -3.

105. $3x^2-10x+2k=0$; product of the roots is -4.

106. $2kx^2+5x-1=0$; product of the roots is $\frac{1}{2}$.

107. $2x^2-kx+9=0$; one root is double the other.

108. $3x^2-4x+k=0$; one root is triple the other.

109. $6x^2-13x+k=0$; one root is the reciprocal of the other.

110. Show that if there is only one real root of the equation $ax^2+bx+c=0$, $a\neq 0$, then $b=\pm 2\sqrt{ac}$.

111. If $x^2\leq 36$, is it true that $x\leq 6$? Prove this or give a counter example.

112. If $x^4\geq 16$, is it true that $x\geq 2$? Prove this or give a counter example.

113. If $x\neq 0$, is $x>1/x$? Prove this or give a counter example.

114. Given a regular polygon having n sides, we can determine the number of diagonals D by the formula

$$D=\frac{n^2-3n}{2}$$

a. If a polygon has 65 diagonals, how many sides does the polygon have?

b. A polygon has 20 diagonals. How many sides does the polygon have?

c. A polygon has n sides. If the number of sides of the polygon is increased by one, by how much is the number of diagonals increased?

115. The sum of the first n positive integers is given by:

$$1+2+3+4+\cdots+(n-1)+n=\frac{n(n+1)}{2}$$

a. The sum of the first n integers is 55. What is n?

b. Find n if the sum of the first n integers is 36.

c. Find $2+3+4+\cdots+n$.

116. The sum of the first n odd positive integers is given by:

$$1+3+5+7+\cdots+(2n-1)=n^2$$

a. Find $5+7+\cdots+(2n-1)$.

b. Find $1+3+5+\cdots+(2n-3)$.

117. The sum of the first n odd positive integers is 121. (See Exercise 116.)

a. Find n.

b. Write out this sum explicitly.

118. The distance d between two points (x_1, y_1) and (x_2, y_2) is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

If the distance between (2, 3) and $(x, 10)$ is 7, find x.

119. The number of two-way links n necessary to create a full-mesh network between x users is given by

$$\frac{x(x-1)}{2} = n$$

How many users can be connected in such a network with 190 links? 4950 links? (See chapter opener.)

120. Solve the following using the quadratic formula, by first storing the discriminant as D, then finding

$$\frac{-b \pm \sqrt{D}}{2a}.$$

$$0.0001x^2 - 0.086x - 48.75 = 0$$

In Exercises 121–128, use Exercises 93–100 and GRAPH the left side of each equation as Y_1, and the right side as Y_2. Then see if the points of intersection verify your algebraic solutions. (You will learn about the theory behind graphing in Chapter 3. For now, just see if the images you generate in the calculator shed light on the problems.)

2.4 Applications of Quadratic Equations

As your knowledge of mathematical techniques and ideas grows, you will become capable of solving an ever wider variety of applied problems. In Section 2.2 we explored many types of word problems that lead to linear equations. We can now tackle a group of applied problems that lead to quadratic equations.

One word of caution: It is possible to arrive at a solution that makes no sense. For example, a negative solution that represents hours worked or the age of an individual is meaningless and must be rejected.

EXAMPLE 1 QUADRATIC EQUATIONS AND WORD PROBLEMS

The larger of two positive numbers exceeds the smaller by 2. If the sum of the squares of the two numbers is 74, find the two numbers.

SOLUTION

If we let

$$x = \text{the larger number}$$

then

$$x - 2 = \text{the smaller number}$$

The sum of the squares of the numbers is 74.

$$(\text{larger number})^2 + (\text{smaller number})^2 = 74$$

$$x^2 + (x-2)^2 = 74$$

$$x^2 + x^2 - 4x + 4 = 74$$
$$2x^2 - 4x - 70 = 0$$
$$x^2 - 2x - 35 = 0$$
$$(x + 5)(x - 7) = 0$$
$$x = 7 \qquad \text{Reject } x = -5.$$

The numbers are then 7 and $(7 - 2) = 5$. Verify that the sum of the squares is indeed 74.

EXAMPLE 2 QUADRATIC EQUATIONS AND WORD PROBLEMS

The length of a pool is 3 times its width, and the pool is surrounded by a grass walk 4 feet wide. If the total area covered and enclosed by the walk is 684 square feet, find the dimensions of the pool.

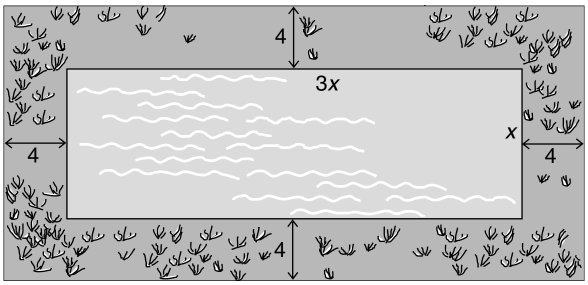


FIGURE 1 See Example 2.

SOLUTION

Drawing diagrams is useful in solving geometric problems. For example, as shown in Figure 1, if we let x = width of the pool, then $3x$ = length of the pool. The region enclosed by the walk has length $3x + 8$ and width $x + 8$. The total area is the product of the length and width, so

$$\text{length} \times \text{width} = 684$$
$$(3x + 8)(x + 8) = 684$$
$$3x^2 + 32x + 64 = 684$$
$$3x^2 + 32x - 620 = 0$$
$$(3x + 62)(x - 10) = 0$$
$$x = 10 \qquad \text{Reject } x = -\frac{62}{3}.$$

The dimensions of the pool are 10 feet by 30 feet.

EXAMPLE 3 QUADRATIC EQUATIONS AND WORD PROBLEMS

Working together, two cranes can unload a ship in 4 hours. The slower crane, working alone, requires 6 hours more than the faster crane to do the job. How long does it take each crane to do the job by itself?

SOLUTION

Let x = number of hours for the faster crane to do the job. Then $x + 6$ = number of hours for the slower crane to do the job. The rate of the faster crane is $\frac{1}{x}$, the portion of the whole job that it completes in 1 hour. Similarly, the rate of the slower crane is $\frac{1}{x+6}$. We display this information in a table.

	Rate	×	Time	=	Work done
Faster crane	$\frac{1}{x}$		4		$\frac{4}{x}$
Slower crane	$\frac{1}{x+6}$		4		$\frac{4}{x+6}$

When the two cranes work together, we must have

$$\begin{pmatrix}\text{work done by}\\ \text{fast crane}\end{pmatrix} + \begin{pmatrix}\text{work done by}\\ \text{slow crane}\end{pmatrix} = 1 \text{ whole job}$$

or

$$\frac{4}{x} + \frac{4}{x+6} = 1$$

To solve, we multiply by the LCD, $x(x + 6)$, obtaining

$$\begin{aligned} 4(x+6) + 4x &= x^2 + 6x \\ 0 &= x^2 - 2x - 24 \\ 0 &= (x+4)(x-6) \\ x = -4 \quad \text{or} \quad x &= 6 \end{aligned}$$

The solution $x = -4$ is rejected because it makes no sense to speak of negative hours of work. Then

$x = 6$ is the number of hours in which the fast crane can do the job alone.

$x + 6 = 12$ is the number of hours in which the slow crane can do the job alone.

Exercise Set 2.4

1. Working together, computers A and B can complete a data-processing job in 2 hours. Computer A working alone can do the job in 3 hours less than computer B working alone. How long does it take each computer to do the job by itself?

2. A graphic designer and her assistant working together can complete an advertising layout in 6 days. The assistant working alone could complete the job in 16 more days than the designer working alone. How long would it take each person to do the job alone?

3. A roofer and his assistant working together can finish a roofing job in 4 hours. The roofer working alone could finish the job in 6 hours less than the assistant working alone. How long would it take each person to do the job alone?

4. A 16-inch by 20-inch mounting board is used to mount a photograph. How wide a uniform border is needed if the photograph occupies $\frac{3}{5}$ of the area of the mounting board?

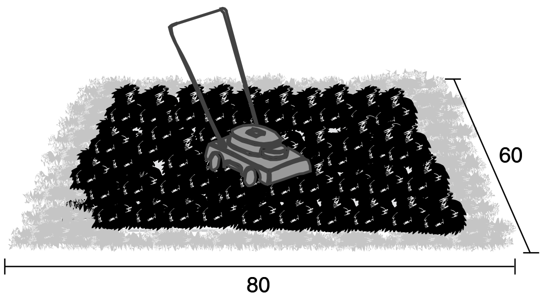
5. The length of a rectangle exceeds twice its width by 4 feet. If the area of the rectangle is 48 square feet, find the dimensions.

6. The length of a rectangle is 4 centimeters less than twice its width. Find the dimensions if the area of the rectangle is 96 square centimeters.

7. The area of a rectangle is 48 square centimeters. If the length and width are each increased by 4 centimeters, the area of the newly formed rectangle is 120 square centimeters. Find the dimensions of the original rectangle.

8. The base of a triangle is 2 feet more than twice its altitude. If the area is 12 square feet, find the dimensions.

9. Find the width of a strip that has been mowed around a rectangular field 60 feet by 80 feet if $\frac{1}{2}$ the lawn has not yet been mowed.



10. The sum of the reciprocals of two consecutive numbers is $\frac{7}{12}$. Find the numbers.

11. The sum of a number and its reciprocal is $\frac{26}{5}$. Find the number.

12. The difference of a number and its reciprocal is $\frac{35}{6}$. Find the number.

13. The smaller of two numbers is 4 less than the larger. If the sum of their squares is 58, find the numbers.

14. The sum of the reciprocals of two consecutive odd numbers is $\frac{8}{15}$. Find the numbers.

15. The sum of the reciprocals of two consecutive even numbers is $\frac{7}{24}$. Find the numbers.

16. A number of students rented a car for \$160 for a one-week camping trip. If another student had joined the original group, each person's share of expenses would have been reduced by \$8. How many students were in the original group?

17. An investor placed an order totaling \$1200 for a certain number of shares of a stock. If the price of each share of stock was \$2 more, the investor would get 30 fewer shares for the same amount of money. How many shares did the investor buy?

18. A fraternity charters a bus for a ski trip at a cost of \$360. When 6 more students join the trip, each person's cost decreases by \$2. How many students were in the original group of travelers?

19. A salesman worked a certain number of days to earn \$192. If he had been paid \$8 more per day, he would have earned the same amount of money in 2 fewer days. How many days did he work?

20. A freelance photographer worked a certain number of days for a newspaper to earn \$480. If she had been paid \$8 less per day, she would have earned the same amount in 2 more days. What was her daily rate of pay?

21. A wire 48 centimeters long is cut into two pieces. Each piece is bent to form a square. Where should the wire be cut so that the sum of the areas of the squares is equal to 80 square centimeters?

22. A circuit has a resistance of 25 ohms and a voltage of 110 volts. The power P in watts when a current I in amperes flows through the circuit is

$$P = 110I - I^2$$

If the power is 121 watts, find the current I.

The following are needed for Exercises 23 and 24: If an object is thrown with an initial speed of v_0, then the distance d the object travels in t seconds is

$$d = 16t^2 + v_0 t$$

if the distance is measured in feet and

$$d = 4.9t^2 + v_0 t$$

if the distance is measured in meters.

23. An object is thrown off the Gateway Arch in St. Louis with an initial velocity of 68 feet per second. The Gateway Arch is 630 feet tall. How long will it take for the object to reach the ground?

24. A warhead fired from an enemy ship in the Persian Gulf is a dud and travels only 100 meters before it hits the water. If it had an initial velocity of 489 meters per second, find the time from the initial launch of the warhead to impact.

25. A principal amount P is invested at a rate of $r \times 100\%$ per year. After 2 years the amount of the investment becomes $S = P(1 + r)^2$. Rachel invests \$7000 at the start of her junior year in high school. What interest rate does her investment require so that she will have \$9100 for her freshman tuition in college 2 years later?

26. The spread of a certain variety of ivy follows the investment model given in Exercise 25, $S = P(1 + r)^2$. Thirty square feet of ivy are planted. Two years later there are 75 square feet of ground cover. At what rate is the plant spreading?

27. According to Einstein's theory, relative to earth time, space travelers will not age as fast as those who remain on earth. In fact, if a space traveler could travel at the speed of light, relative to those on earth, the traveler would not age at all. The relationship between time on earth t_e, and time in space t_s is given by

$$t_s = t_e \sqrt{1 - \frac{v^2}{c^2}}$$

where v is the velocity of the space traveler and c is the speed of light. One of two brothers makes a round trip to Arcturus. The elapsed time of the trip for the brother on earth is 80.8 years, whereas the elapsed time for the traveling brother is 11.4 years. If the traveling brother maintained a constant speed for the entire trip, find this speed in terms of c.

28. There exists a gravitational force of attraction between all particles. Newton's law of gravitation defines the relationship between the force F exerted by a particle of mass m_1 on another particle of mass m_2

when the distance between the two particles is r.

$$F = \frac{Gm_1m_2}{r^2}$$

where $G = 6.67 \times 10^{-11}$ Nm^2/kg^2 is the universal gravitational constant. (N is the abbreviation for Newtons, a measure of force.) The gravitational force that attracts a 65-kilogram boy to a 50-kilogram girl is 8.67×10^{-7} N. How many meters apart are the boy and the girl?

29. Angular displacement θ is defined by:

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha_0 t^2$$

where θ_0, ω_0 and α_0 are the angular displacement, angular velocity, and angular acceleration, respectively, all at time $t = 0$.

a. Let $\theta = 11$ radians, $\theta_0 = 1$ radian, $\omega_0 = 3$ radians per second and $\alpha_0 = 2$ radians per second squared. Solve for t.

b. Find a general formula for t by using the quadratic formula.

30. The equilibrium point in economic theory is that price where demand equals supply. For the following supply and demand equations, find the equilibrium point.

a. $d = \dfrac{1500}{p}, \quad s = 500p - 250$

b. $d = \dfrac{1400}{p}, \quad s = 1200p - 3800$

31. In probability theory, a binomial random variable x can be approximated by a normal random variable z by the equation

$$z = \frac{x - np}{\sqrt{np(1-p)}}$$

where n is the number of trials in a binomial experiment and p is the probability of success in one trial of the experiment. If the number of trials is 16, $x = 1$ and $z = -2.48$, what is p?

32. An oil company has decided to replace two old cylindrical storage tanks with one new cylindrical storage tank constructed from a material guaranteed to keep the oil at a constant temperature. The old tanks were both 25 feet high, however, one tank had a radius of 12 feet whereas the other had a radius of 16 feet. The new tank will also be 25 feet high. Find the radius of the new tank if it is to hold the same amount of oil as both of the old tanks. (*Hint:* The volume of a cylinder is $V = \pi r^2 h$, where r is the radius and h is the height.)

33. The area of a circle is 10π. What is the radius?

34. The surface area of a cube is 294 square inches. What is the length of each edge of the cube?

35. It is believed that the most visually pleasing rectangle with length L and width W satisfies the following equation

$$\frac{L + W}{L} = \frac{L}{W}$$

a. What is L if $W = 5$?

b. Solve this equation for W in terms of L.

2.5 Linear and Quadratic Inequalities

Much of the terminology of equations carries over to inequalities. A **solution of an inequality** is a value of the unknown that satisfies the inequality, and the solution set is composed of all solutions. The properties of inequalities listed in Section 1.2 enable us to use the same procedures in solving inequalities as in solving equations *with one exception.*

Multiplication or division of an inequality by a negative number reverses the direction of the inequality.

We will concentrate for now on solving a **linear inequality**, that is, an inequality in which the unknown appears only in the first degree.

EXAMPLE 1 LINEAR INEQUALITIES

Solve the inequality $2x + 11 \geq 5x - 1$.

SOLUTION

We perform addition and subtraction for inequalities to collect terms in x just as we did for equations.

$$2x + 11 \geq 5x - 1$$
$$2x \geq 5x - 12$$
$$-3x \geq -12$$

We now divide both sides of the inequality by -3, a negative number, and therefore reverse the sense of the inequality.

$$\frac{-3x}{-3} \leq \frac{-12}{-3}$$
$$x \leq 4$$

✔ Progress Check

Solve the inequality $3x - 2 \geq 5x + 4$.

Answer

$x \leq -3$

WARNING

Given the inequality

$$-2x \geq -6$$

it is a common error to conclude that dividing by -2 gives $x \leq -3$. Multiplication or division by a negative number changes the sense of the inequality, but the *signs* obey the usual rules of algebra. Thus,

$$-2x \geq -6$$
$$\frac{-2x}{-2} \leq \frac{-6}{-2} \qquad \text{Reverse sense of the inequality.}$$
$$x \leq 3$$

There are three methods commonly used to describe subsets of the real numbers: graphs on a real number line, interval notation, and set notation. Since there will be occasions when we want to use each of these schemes, this is a convenient time to introduce them and to apply them to inequalities.

The **graph of an inequality** is the set of all points satisfying the inequality. The graph of the inequality $a \le x < b$ is shown in Figure 2. The portion of the real number line that is in bold is the solution set of the inequality. The circle at point a is filled in to indicate that a is also a solution of the inequality; the circle at point b is left open to indicate that b is not a member of the solution set.

FIGURE 2 Graph of $a \le x < b$

An **interval** is a set of numbers on the real number line that forms a line segment, a half line, or the entire real number line. The subset shown in Figure 2 is written in **interval notation** as $[a, b)$, where a and b are the **endpoints** of the interval. A bracket, [or], indicates that the endpoint is included, and a parenthesis, (or), indicates that the endpoint is not included. The interval $[a, b]$ is called a **closed interval** because both endpoints are included. The interval (a, b) is called an **open interval** because neither endpoint is included. Finally, the intervals $[a, b)$ and $(a, b]$ are called **half-open intervals.**

The set of all real numbers satisfying a given property P is written as

$$\{x \mid x \text{ satisfies property } P\}$$

which is read as "the set of all x such that x satisfies property P." This form, called **set notation**, provides a third means of designating subsets of the real number line. Thus, the interval $[a, b)$ shown in Figure 1 is written as

$$\{x \mid a \le x < b\}$$

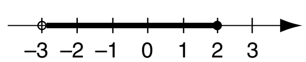
which indicates the x must satisfy the inequalities $x \ge a$ and $x < b$.

EXAMPLE 2 GRAPHS OF FINITE INTERVALS

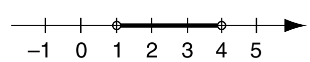
Graph each of the given intervals on a real number line and indicate the same subset of the real number line in set notation.

a. $(-3, 2]$ b. $(1, 4)$ c. $[-4, -1]$

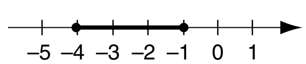
SOLUTION

a. 

$\{x \mid -3 < x \le 2\}$

b. 

$\{x \mid 1 < x < 4\}$

c. 

$\{x \mid -4 \le x \le -1\}$

To describe the inequality $x > 2$ or the inequality $x \leq 3$ in interval notation, we need to introduce the symbols ∞ and $-\infty$, read "**infinity**" and "minus infinity," respectively. The inequality $x > 2$ is then written as $(2, \infty)$, and the inequality $x \leq 3$ is written as $(-\infty, 3]$. They are graphed on a real number line as shown in Figure 3. Note that ∞ and $-\infty$ are symbols, not numbers, indicating that the intervals extend indefinitely. An interval using one of these symbols is called an **infinite interval**. The interval $(-\infty, \infty)$ designates the entire real number line. Square brackets must never be used around ∞ and $-\infty$ since they are not real numbers.

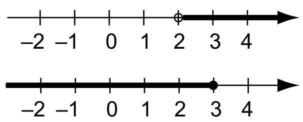


FIGURE 3 Graphs of Infinite Intervals

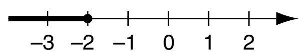
If the endpoint of the graph of an inequality is *not* specifically identified by an *open circle* or by a *filled-in circle,* then we assume that the graph continues forever in the direction where the endpoint is "missing." (See Figure 3.) (There are some texts that indicate that a graph continues forever in a particular direction by placing an arrow on the graph pointing in that direction. We shall *not* use this "arrow" notation in this text.)

EXAMPLE 3 GRAPHING AND SOLVING LINEAR INEQUALITIES

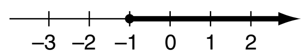
Graph each inequality and write the solution set in interval notation.

a. $x \leq -2$ b. $x \geq -1$ c. $x < 3$

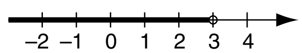
SOLUTION

a. 

$(-\infty, -2]$

b. 

$[-1, \infty)$

c. 

$(-\infty, 3)$

EXAMPLE 4 GRAPHING AND SOLVING LINEAR INEQUALITIES

Solve the inequality.

$$\frac{x}{2} - 9 < \frac{1 - 2x}{3}$$

Graph the solution set, and write the solution set in both interval notation and set notation.

SOLUTION

To clear the inequality of fractions, we multiply both sides by the LCD of all fractions, which is 6.

$$3x - 54 < 2(1 - 2x)$$
$$3x - 54 < 2 - 4x$$
$$7x < 56$$
$$x < 8$$

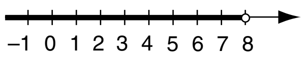


FIGURE 4 Graph of the Solution Set of Example 4

We may write the solution set as $\{x \mid x < 8\}$ or as the infinite interval $(-\infty, 8)$. The graph of the solution set is shown in Figure 4.

EXAMPLE 5 LINEAR INEQUALITIES

Solve the inequalities.

a. $\dfrac{2(x+1)}{3} < \dfrac{2x}{3} - \dfrac{1}{5}$ b. $2(x-1) < 2x + 5$

SOLUTION

a. The LCD of all fractions is 15. Multiplying both sides of the inequality by 15, we obtain

$$10(x+1) < 10x - 3$$
$$10x + 10 < 10x - 3$$
$$10 < -3$$

Our procedure has led to a contradiction, indicating that there is no solution to the inequality.

b. Expanding and simplifying leads to the inequality

$$-2 < 5$$

Since this inequality is true for all real values of x, we conclude that the solution set is the set of all real numbers.

✔ Progress Check

Solve, and write the answers in interval notation.

a. $\dfrac{3x-1}{4} + 1 > 2 + \dfrac{x}{3}$ b. $\dfrac{2x-3}{2} \geq x + \dfrac{2}{5}$

Answers

a. $(3, \infty)$ b. no solution

EXAMPLE 6 INEQUALITIES AND WORD PROBLEMS

A taxpayer may choose to pay a 20% tax on the gross income or a 25% tax on the gross income less \$4000. Above what income level should the taxpayer elect to pay at the 20% rate?

SOLUTION

If we let x = gross income, then the choice available to the taxpayer is

a. pay at the 20% rate on the gross income, that is, pay $0.20x$, or

b. pay at the 25% rate on the gross income less \$4000, that is, pay

$$0.25(x - 4000)$$

To determine when (a) produces a lower tax than (b), we must solve

$$\begin{aligned} 0.20x &< 0.25(x - 4000) \\ 0.20x &< 0.25x - 1000 \\ -0.05x &< -1000 \\ x &> \frac{-1000}{-0.05} = 20{,}000 \end{aligned}$$

The taxpayer should choose to pay at the 20% rate if the gross income is more than \$20,000.

✔ Progress Check

A customer is offered the following choice of telephone services: unlimited local calls at a fixed \$20 monthly charge, or a base rate of \$8 per month plus \$0.06 per message unit. At what level of use does it cost less to choose the unlimited service?

Answer

Unlimited service costs less when the anticipated use exceeds 200 message units.

Compound Inequalities

We can solve compound inequalities such as

$$1 < 3x - 2 \leq 7$$

by operating on both inequalities at the same time.

$3 < 3x \leq 9$ Add $+2$ to each member.

$1 < x \leq 3$ Divide each member by 3.

The solution set is the half-open interval $(1, 3]$.

Note that the statement of the compound inequality

$$1 < 3x - 2 \leq 7$$

actually represents three inequalities:

$$\begin{aligned} 1 &< 3x - 2 \\ 3x - 2 &\leq 7 \\ 1 &\leq 7 \end{aligned}$$

EXAMPLE 7 COMPOUND INEQUALITIES

Solve the inequality $-3 \le 1 - 2x < 6$, and write the answer in interval notation.

SOLUTION

Operating on this inequality, we have

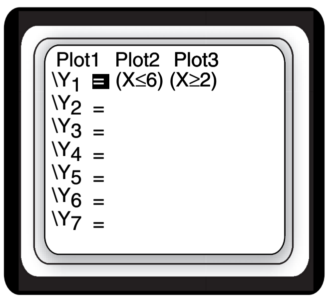
$$-4 \le -2x < 5 \qquad \text{Add } -1 \text{ to each member.}$$

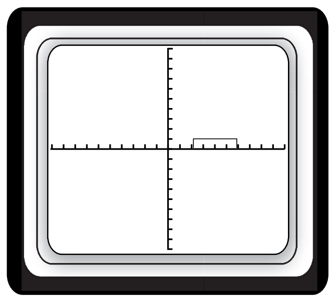
$$2 \ge x > -\frac{5}{2} \qquad \text{Divide each member by } -2.$$

The solution set is the half-open interval $\left(-\frac{5}{2}, 2\right]$.

Graphing Calculator Alert

With many graphing calculators, you can use the GRAPH option to visualize the number-line solution to a linear inequality. The x-axis on the graph display represents the number line, and a horizontal line just above it represents the solution. Try by letting $Y_1 = (x \le 6)(x \ge 2)$.





✔ Progress Check

Solve the inequality $-5 < 2 - 3x < -1$, and write the answer in interval notation.

Answer

$\left(1, \frac{7}{3}\right)$

Critical Value Method

The **Critical Value Method** is an alternative approach to solving inequalities. In fact, we shall be relying upon this method throughout the remainder of the text.

> The **critical values** of an inequality are
>
> 1. those values for which either side of the inequality is not defined (such as a denominator equal to 0),
> 2. those values that are solutions to the equation obtained by replacing the inequality sign with an equal sign.

The critical values determine endpoints of intervals on the real number line. The inequality in question either satisfies all points in a given interval, or no points in a given interval. In order to find out in which intervals the inequality holds, we may test *any* point from each interval. We call such points **test points**. We follow this technique using Example in Table 4.

EXAMPLE 8 RATIONAL EXPRESSION INEQUALITIES

Solve the inequality.

$$\frac{x+1}{x-1} \geq 2$$

SOLUTION

The inequality is not defined where $x - 1 = 0$, that is, where $x = 1$. Solving the equation

$$\frac{x+1}{x-1} = 2$$

$$x + 1 = 2x - 2$$

$$x = 3$$

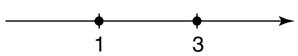


FIGURE 5 Critical Values for Example 8

Therefore, the critical values are 1 and 3 as shown in Figure 5.

TABLE 4 Solving Inequalities by the Critical Value Method

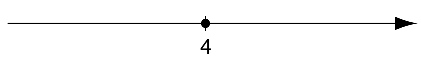
Method	Example: $2x + 11 \geq 5x - 1$
Step 1. Find the critical values of the inequality. a. values where the inequality is not defined b. Replace the inequality sign by an equal sign and solve.	*Step 1.* a. Both sides of the inequality are defined everywhere. b. $2x + 11 = 5x - 1$ $12 = 3x$ $x = 4$
Step 2. Plot the critical values on the real number line.	*Step 2.* 

TABLE 4 Solving Inequalities by the Critical Value Method (cont.)

Method	Example: $2x + 11 \geq 5x - 1$	
Step 3. Try a test point in each interval, and try each critical value.	*Step 3.* Interval: $x < 4$ Test point: $x = 0$	
	$2(0) + 11 \geq 5(0) - 1$	True
	Critical Value: $x = 4$ Test point: $x = 4$	
	$2(4) + 11 \geq 5(4) - 1$	True
	Interval: $x > 4$ Test point: $x = 5$	
	$2(5) + 11 \geq 5(5) - 1$	False
Step 4. Find the solution.	*Step 4.* $x \leq 4$	

SOLUTION BY THE CRITICAL VALUE METHOD

Interval, Critical Value	Test Point	Substitution	Verification
$x < 1$	$x = 0$	$\dfrac{0+1}{0-1} \geq 2$	False
$x = 1$	$x = 1$	$\dfrac{1+1}{1-1} \geq 2$	False
$1 < x < 3$	$x = 2$	$\dfrac{2+1}{2-1} \geq 2$	True
$x = 3$	$x = 3$	$\dfrac{3+1}{3-1} \geq 2$	True
$x > 3$	$x = 4$	$\dfrac{4+1}{4-1} \geq 2$	False

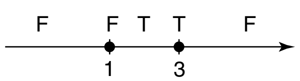


FIGURE 6 Summary of Critical Value Analysis

These results are summarized in Figure 6. Therefore, the solution set consists of all real numbers

$$\{x \mid 1 < x \leq 3\}$$

Second-Degree Inequalities

The Critical Value Method can also be applied to **second-degree inequalities.** This requires the solution of a quadratic equation rather than a linear equation.

EXAMPLE 9 QUADRATIC INEQUALITIES

Solve the inequality $x^2 - 2x > 15$ and graph the solution.

SOLUTION

The inequality is defined everywhere. Solving the equation

$$\begin{aligned} x^2 - 2x &= 15 \\ x^2 - 2x - 15 &= 0 \\ (x + 3)(x - 5) &= 0 \\ x &= -3, 5 \end{aligned}$$

Therefore, the critical values are -3 and 5 as shown in Figure 7.

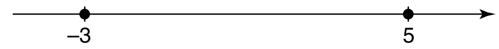


FIGURE 7 Critical Values for Example 9

SOLUTION BY THE CRITICAL VALUE METHOD

Interval, Critical Value	Test Point	Substitution	Verification
$x < -3$	$x = -5$	$(-5)^2 - 2(-5) > 15$	True
$x = -3$	$x = -3$	$(-3)^2 - 2(-3) > 15$	False
$-3 < x < 5$	$x = 0$	$0^2 - 2(0) > 15$	False
$x = 5$	$x = 5$	$(5)^2 - 2(5) > 15$	False
$x > 5$	$x = 6$	$(6)^2 - 2(6) > 15$	True

These results are summarized in Figure 8.

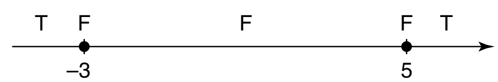


FIGURE 8 Summary of Critical Value Analysis

Therefore, the solution set is

$$\{x \mid x < -3 \quad \text{or} \quad x > 5\}$$

which consists of the real numbers in the open intervals $(-\infty, -3)$ and $(5, \infty)$. The solution set is shown in Figure 9.

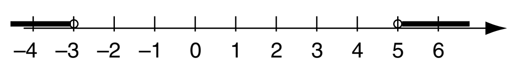


FIGURE 9 Graph of Solution Set for Example 9

✔ Progress Check

Solve the inequalities:

a. $2x^2 \geq 5x + 3$ b. $\dfrac{2x - 3}{1 - 2x} \geq 0$

Answers

a. $\left\{x \,\middle|\, x \leq -\dfrac{1}{2} \quad \text{or} \quad x \geq 3\right\}$ b. $\left\{x \,\middle|\, \dfrac{1}{2} < x \leq \dfrac{3}{2}\right\}$

EXAMPLE 10 POLYNOMIAL INEQUALITIES

Solve the inequality $(x - 2)(2x + 5)(3 - x) < 0$.

SOLUTION

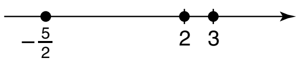


FIGURE 10 Critical Values for Example 10

The inequality is defined everywhere. Solving the equation

$$(x - 2)(2x + 5)(3 - x) = 0$$

$$x = 2, -\frac{5}{2}, 3$$

Therefore the critical values are $-\frac{5}{2}$, 2, and 3 as shown in Figure 10.

SOLUTION BY THE CRITICAL VALUE METHOD

Interval, Critical Value	Test Point	Substitution	Verification
$x < -\frac{5}{2}$	$x = -3$	$(-3 - 2)(2(-3) + 5)(3 - (-3)) < 0$	False
$x = -\frac{5}{2}$	$x = -\frac{5}{2}$	$\left(-\frac{5}{2} - 2\right)\left(2\left(-\frac{5}{2}\right) + 5\right)\left(3 - \left(-\frac{5}{2}\right)\right) < 0$	False
$-\frac{5}{2} < x < 2$	$x = 0$	$(0 - 2)(2(0) + 5)(3 - 0) < 0$	True
$x = 2$	$x = 2$	$(2 - 2)(2(2) + 5)(3 - 2) < 0$	False
$2 < x < 3$	$x = \frac{5}{2}$	$\left(\frac{5}{2} - 2\right)\left(2\left(\frac{5}{2}\right) + 5\right)\left(3 - \frac{5}{2}\right) < 0$	False
$x = 3$	$x = 3$	$(3 - 2)(2(3) + 5)(3 - 3) < 0$	False
$x > 3$	$x = 4$	$(4 - 2)(2(4) + 5)(3 - 4) < 0$	True

These results are summarized in Figure 11.

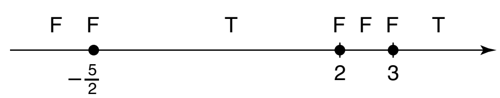


FIGURE 11 Summary of Critical Value Analysis

Therefore, the solution set consists of all real numbers

$$\{x \mid -\frac{5}{2} < x < 2 \quad \text{or} \quad x > 3\}$$

which consists of the real numbers in the open intervals $(-\frac{5}{2}, 2)$, $(3, \infty)$.

✔ Progress Check

Solve the inequality $(2y - 9)(6 - y)(y + 5) \geq 0$.

Answers

$\left\{y \mid y \leq -5 \quad \text{or} \quad \frac{9}{2} \leq y \leq 6\right\}$ or $(-\infty, -5], \left[\frac{9}{2}, 6\right]$

EXAMPLE 11 QUADRATIC INEQUALITIES

Solve the inequality $x^2 + 1 > 0$.

SOLUTION

The inequality is defined everywhere. The equation $x^2 = -1$ has no real roots. Thus, we have no critical values, and there is only one interval, namely, the entire real number line. If we choose our test point to be 0, we have

$$0^2 > -1$$

which is true. Therefore, the solution set consists of all real numbers.

Exercise Set 2.5

In Exercises 1–9, express the given inequality in interval notation.

1. $-5 \leq x < 1$
2. $-4 < x \leq 1$
3. $x > 9$
4. $x \leq -2$
5. $-12 \leq x \leq -3$
6. $x \geq -5$
7. $3 < x < 7$
8. $x < 17$
9. $-6 < x \leq -4$

In Exercises 10–18, express the given interval as an inequality.

10. $(-4, 3]$
11. $[5, 8]$
12. $(-\infty, -2]$
13. $(3, \infty)$
14. $[-3, 10)$
15. $(-\infty, 5]$
16. $(-2, -1)$
17. $[0, \infty)$
18. $(-5, 7)$

In Exercises 19–36, solve the inequality and graph the result.

19. $x + 4 < 8$
20. $x + 5 < 4$
21. $x + 3 < -3$
22. $x - 2 \leq 5$
23. $x - 3 \geq 2$
24. $x + 5 \geq -1$
25. $2 < a + 3$
26. $-5 > b - 3$
27. $2y < -1$
28. $3x < 6$
29. $2x \geq 0$
30. $-\frac{1}{2}y \geq 4$
31. $2r + 5 < 9$
32. $3x - 2 > 4$
33. $3x - 1 \geq 2$
34. $\frac{-1}{2x + 3} > 0$
35. $\frac{4}{5 - 3x} < 0$
36. $\frac{3}{3x - 1} > 0$

Solve the given inequality in Exercises 37–60, and write

37. $4x + 3 \leq 11$
38. $\frac{1}{2}y - 2 \leq 2$
39. $\frac{3}{2}x + 1 \geq 4$
40. $-5x + 2 > -8$
41. $4(2x + 1) < 16$
42. $3(3r - 4) \geq 15$
43. $2(x - 3) < 3(x + 2)$
44. $4(x - 3) \geq 3(x - 2)$
45. $3(2a - 1) > 4(2a - 3)$

46. $2(3x - 1) + 4 < 3(x + 2) - 8$

47. $\frac{2}{3}(x + 1) + \frac{5}{6} \geq \frac{1}{2}(2x - 1) + 4$

48. $\frac{1}{4}(3x + 2) - 1 \leq -\frac{1}{2}(x - 3) + \frac{3}{4}$

49. $\frac{x - 1}{3} + \frac{1}{5} < \frac{x + 2}{5} - \frac{1}{3}$

50. $\frac{x}{5} - \frac{1 - x}{2} > \frac{x}{2} - 3$

51. $3(x + 1) + 6 \geq 2(2x - 1) + 4$

52. $4(3x + 2) - 1 \leq -2(x - 3) + 15$

53. $-2 < 4x \leq 5$

54. $3 \leq 6x < 12$

55. $-4 \leq 2x + 2 \leq -2$

56. $5 \leq 3x - 1 \leq 11$

57. $3 \leq 1 - 2x < 7$

58. $5 < 2 - 3x \leq 11$

59. $-8 < 2 - 5x \leq 7$

60. $-10 < 5 - 2x < -5$

In Exercises 61–67, translate from words to an algebraic problem and solve.

61. A student has grades of 42 and 70 on the first two tests of the semester. If an average of 70 is required to obtain a C grade, what is the minimum score the student must achieve on the third exam to obtain a C?

62. A compact car can be rented from firm A for \$160 per week with no charge for mileage or from firm B for \$100 per week plus 20 cents for each mile driven. If the car is driven m miles, for what values of m does it cost less to rent from firm A?

63. An appliance salesperson is paid \$30 per day plus \$25 for each appliance sold. How many appliances must be sold for the salesperson's income to exceed \$130 per day?

64. A pension trust invests \$6000 in a bond that pays 5% simple interest per year. Additional funds are to be invested in a more speculative bond paying 9% simple interest per year, so that the return on the total investment will be at least 6%. What is the minimum amount that must be invested in the more speculative bond?

65. A book publisher spends \$38,000 on editorial expenses and \$12 per book for manufacturing and sales expenses in the course of publishing a psychology textbook. If the book sells for \$25, how many copies must be sold to show a profit?

66. If the area of a right triangle is not to exceed 80 square inches and the base is 10 inches, what values may be assigned to the altitude h?

67. A total of 70 meters of fencing material is available with which to enclose a rectangular area. If the width of the rectangle is 15 meters, what values can be assigned to the length L?

In Exercises 68–95, indicate the solution set of each inequality on a real number line.

68. $x^2 + 5x + 6 > 0$

69. $x^2 + 3x - 4 \leq 0$

70. $2x^2 - x - 1 < 0$

71. $3x^2 - 4x - 4 \geq 0$

72. $4x - 2x^2 < 0$

73. $r^2 + 4r \geq 0$

74. $\frac{x + 5}{x + 3} \leq 0$

75. $\frac{x - 6}{x + 4} \geq 0$

76. $\frac{2r + 1}{r - 3} \leq 0$

77. $\frac{x - 1}{2x - 3} \geq 0$

78. $\frac{3s + 2}{2s - 1} \geq 0$

79. $\frac{4x + 5}{x^2} \leq 0$

80. $(x + 2)(3x - 2)(x - 1) > 0$

81. $(x - 4)(2x + 5)(2 - x) \leq 0$

82. $x^2 + x - 6 > 0$

83. $x^2 - 3x - 10 \geq 0$

84. $2x^2 - 3x - 5 < 0$

85. $3x^2 - 4x - 4 \leq 0$

86. $\frac{2r + 3}{2r - 1} < 0$

87. $\frac{3x + 2}{2x - 3} \geq 0$

88. $\frac{x - 1}{x + 1} \geq 0$

89. $\frac{2x - 1}{x + 2} \leq 0$

90. $6x^2 + 8x + 2 \geq 0$

91. $2x^2 + 5x + 2 \leq 0$

92. $(y - 3)(2 - y)(2y + 4) \geq 0$

93. $(2x + 5)(3x - 2)(x + 1) < 0$

94. $(x - 3)(1 + 2x)(3x + 5) > 0$

95. $(1 - 2x)(2x + 1)(x - 3) \le 0$

In Exercises 96–99, find the values of x for which the given expression has real values.

96. $\sqrt{(x-2)(x+1)}$ 97. $\sqrt{(2x+1)(x-3)}$

98. $\sqrt{2x^2 + 7x + 6}$ 99. $\sqrt{2x^2 + 3x + 1}$

100. A manufacturer of solar heaters finds that when x units are made and sold, the profit (in thousands of dollars) is given by $x^2 - 50x - 5000$. For what values of x will the firm show a loss?

101. A ball thrown directly upward from ground level at an initial velocity of 40 feet per second attains a height d given by $d = 40t - 16t^2$ after t seconds. During what time interval is the ball at a height of at least 16 feet?

102. A rectangle has length x and width $x - 4$.
 a. Find the inequality that states that the perimeter of the rectangle must be at least 24 units, and solve for x.
 b. Find the inequality that states that the area of the rectangle must be less than 12 square units, and solve for x.

103. Each of the two congruent sides of an isosceles triangle are 10 centimeters more than $\frac{1}{2}$ the length of the base. If the perimeter of the triangle is to be at most 100 centimeters, what is the maximum length of the base?

104. Morry's best time in the 70-meter track event is 9.5 seconds. Rob wants to beat Morry's best time. He runs the first half of the event at a speed of 7 meters per second. What is the maximum time Rob has left to run the second half of the event?

105. A carpet factory manufactures bolts of carpet 10 feet wide. A large bolt of carpet covers 8 linear feet more than a small bolt of carpet. If the large bolt of carpet covers at most 200 square feet of floor, what is the largest length of a small bolt?

106. Charles and Morry face each other at opposite ends of an 880-meter track. Charles runs this distance at 420 meters per minute, and Morry runs the same distance at 300 meters per minute. At the sound of the gun, the boys start running toward each other. Charles always arrives at a point P on the track before Morry. What is the farthest distance P could be from Charles's end of the track?

107. The power P in watts, total resistance R in ohms, and current I in amperes of a circuit are related by the equation

$$P = I^2R$$

The power output can be at most 1200 watts. What is the maximum current in the circuit if the total resistance is 48 ohms?

108. A string on a musical instrument has frequency f (hertz or vibrations per second) and is defined by

$$f = \frac{1}{2L}\sqrt{\frac{10^5FL}{m}}$$

where F is the tension of the string in Newtons, L is the length of the string in centimeters and m is the mass of the string in grams. The D string on a violin has a length of 45 centimeters and a mass of 0.9 grams. What is the maximum frequency of the D string if the tension of the string can be at most 150 Newtons?

109. Suppose the pressure P in pounds and the volume V in cubic inches of a confined gas is

$$P = \frac{150}{V}$$

What range of volume corresponds to a range of pressure between 50 pounds per square inch and 75 pounds per square inch?

110. Marilyn called Y's Buy Oil company to have her oil tank filled. She was not exactly sure how many gallons of oil were needed. She was told that oil would cost \$2.079 per gallon if she purchased 150 gallons or

more, and would cost \$2.179 per gallon if she purchased less than 150 gallons. For what number of gallons is it more cost effective for Marilyn to buy 150 gallons than to buy less than 150 gallons?

111. Michael earned 310 points before the final exam in his precalculus course. He must have at least 80% of a total of 600 points to get a B in this class. The final is worth 200 points. What is the lowest possible score Michael can get on his final exam and still get a B?

112. An economist hired by the Hiccup Seed Company has found that the company's profit, in hundred thousands of dollars, is

$$P = 6x^2 - 70x + 50$$

where x is the amount, in thousands, of seed packets sold. For what values of x does the Hiccup Seed Company make a profit?

113. The relationship between degrees Celsius and degrees Fahrenheit is given by

$$F = \frac{9}{5}C + 32$$

What temperature range in °F corresponds to –10°C to 20°C?

114. For what values of a is $a^2 > a$, provided $a > 0$?

115. Anthropologists use a ratio called the cephalic index in order to classify different genetic groupings. The cephalic index C requires that you measure the width W and length L of the top of a person's head. The index is as follows

$$C = \frac{100W}{L}$$

A Native American tribe in Oklahoma has a cephalic index of 67 ± 1. If the average length of their heads is 10 inches, what is the smallest and largest width of the heads of the members of this tribe?

116. Suppose you are setting up a full-mesh network for x users; and n, the number of two-way connections required to link all users pairwise, must be no greater than 132. For what range of x values can you set up your network?

$$\frac{x(x-1)}{2} = n$$

In Exercises 117–144, GRAPH the left side of each inequality in Exercises 68–95. Then find the ZEROS and use this information to verify the solutions you obtained algebraically. (Graphing will be described more fully in Chapter 3. If necessary, look ahead to grasp the basics of the technique required.)

145. Graph the left side of the inequality below as Y_1, and the right side as Y_2. Your graphing calculator may be able to locate the points of intersection. Find the solution interval, rounding if necessary.

$$\frac{x-1}{3x+2} \le 1$$

146. ***Mathematics in Writing:*** Write a three-paragraph essay. In the first paragraph, describe in your own words how to solve a quadratic inequality algebraically. In the second paragraph, describe how to solve it graphically. In the final paragraph, compare the two methods and state which one you prefer.

2.6 Absolute Value in Equations and Inequalities

In Section 1.2, we discussed the use of absolute value notation to indicate distance, and we provided this formal definition

$$|x| = \begin{cases} x \text{ when } x \geq 0 \\ -x \text{ when } x < 0 \end{cases}$$

The following example illustrates the application of this definition to the solution of equations involving absolute value.

EXAMPLE 1 ABSOLUTE VALUE IN EQUATIONS

Solve the equation $|2x - 7| = 11$.

SOLUTION

We apply the definition of absolute value and consider two cases.

Case 1. $2x - 7 \geq 0$

With the first part of the definition,

$$|2x - 7| = 2x - 7 = 11$$
$$2x = 18$$
$$x = 9$$

Case 2. $2x - 7 < 0$

With the second part of the definition,

$$|2x - 7| = -(2x - 7) = 11$$
$$-2x + 7 = 11$$
$$x = -2$$

Alternatively, we can solve $2x - 7 = 11$ to obtain $x = 9$ and $2x - 7 = -11$ to obtain $x = -2$.

✔ Progress Check

Solve each equation and check the solution(s).

a. $|x + 8| = 9$ b. $|3x - 4| = 7$

Answers

a. $1, -17$ b. $\frac{11}{3}, -1$

When used in inequalities, absolute value notation plays an important and frequently used role in higher mathematics. To solve inequalities involving absolute value, we recall that $|x|$ is the distance between the origin and the point on the real number line corresponding to x. For $a > 0$, the solution set of the inequality $|x| < a$ is then seen to consist of all real numbers whose distance from the origin is less than a, that is, all real numbers in the open interval $(-a, a)$, shown in Figure 12. Similarly, if $|x| > a > 0$, the solution set consists of all real numbers whose distance from the origin is greater than a, that is, all points

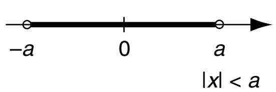


FIGURE 12 Graph of Solution Set for $|x| < a$

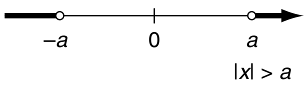


FIGURE 13 Graph of Solution Set for $|x| > a$

in the two infinite intervals $(-\infty, -a)$ and (a, ∞), shown in Figure 13. Of course, $|x| \le a$ and $|x| \ge a$ include the endpoints a and $-a$, and the circles are filled in. An alternative statement of the solution set is

$$\{x \mid x < -a \quad \text{or} \quad x > a\}$$

We will use the Critical Value Method for solving inequalities involving the absolute value in a manner similar to that found in Section 2.5.

EXAMPLE 2 ABSOLUTE VALUE IN INEQUALITIES

Solve the inequality $2|2x - 5| \le 14$. Write the solution set in interval notation and graph the solution.

SOLUTION

The inequality is defined everywhere. Solving the equation

$$\begin{aligned} 2|2x - 5| &= 14 &\quad \text{or} \quad && |2x - 5| &= 7 \\ 2x - 5 &= 7 &\quad \text{or} \quad && 2x - 5 &= -7 \\ x &= 6 &\quad \text{or} \quad && x &= -1 \end{aligned}$$

Therefore, the critical values are -1 and 6, as shown in Figure 14.

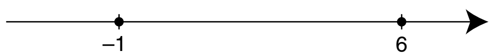


FIGURE 14 Critical Values for Example 2

SOLUTION BY THE CRITICAL VALUE METHOD

Interval, Critical Value	Test Point	Substitution	Verification
$x < -1$	$x = -2$	$2\|2(-2) - 5\| \le 14$	False
$x = -1$	$x = -1$	$2\|2(-1) - 5\| \le 14$	True
$-1 < x < 6$	$x = 0$	$2\|2(0) - 5\| \le 14$	True
$x = 6$	$x = 6$	$2\|2(6) - 5\| \le 14$	True
$x > 6$	$x = 7$	$2\|2(7) - 5\| \le 14$	False

These results are summarized in Figure 15.

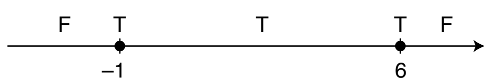


FIGURE 15 Summary of Critical Value Analysis

Therefore, the solution set consists of all real numbers in the closed interval $[-1, 6]$, and the graph of the solution set is shown in Figure 16.

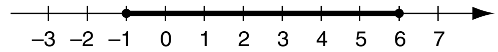


FIGURE 16 Graph of Solution Set for Example 2

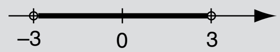
✔ Progress Check

Solve each inequality, graph the solution set, and write the solution set in interval notation.

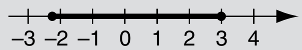
a. $|x| < 3$ b. $|3x - 1| \leq 8$ c. $|x| < -2$

Answers

a. $(-3, 3)$



b. $\left[-\frac{7}{3}, 3\right]$



c. There is no solution since $|x|$ is always nonnegative and thus cannot be less than -2.

EXAMPLE 3 ABSOLUTE VALUE IN INEQUALITIES

Solve the inequality $|2x - 6| > 4$, write the solution set in interval notation, and graph the solution.

SOLUTION

The inequality is defined everywhere. Solving the equation

$$|2x - 6| = 4$$

$$2x - 6 = 4 \quad \text{or} \quad 2x - 6 = -4$$

$$x = 5 \quad \text{or} \quad x = 1$$

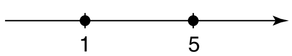


FIGURE 17 Critical Values for Example 3

Therefore, the critical values are 1 and 5 as shown in Figure 17.

SOLUTION BY THE CRITICAL VALUE METHOD

Interval, Critical Value	Test Point	Substitution	Verification
$x < 1$	$x = 0$	$\|2(0) - 6\| > 4$	True
$x = 1$	$x = 1$	$\|2(1) - 6\| > 4$	False
$1 < x < 5$	$x = 2$	$\|2(2) - 6\| > 4$	False
$x = 5$	$x = 5$	$\|2(5) - 6\| > 4$	False
$x > 5$	$x = 6$	$\|2(6) - 6\| > 4$	True

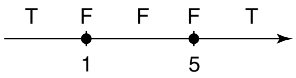


FIGURE 18 Summary of Critical Value Analysis

These results are summarized in Figure 18. Therefore, the solution set consists of all real numbers in the infinite intervals $(-\infty, 1)$ and $(5, \infty)$. The graph of the solution set is shown in Figure 19.

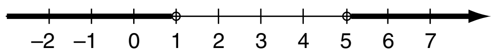


FIGURE 19 Graph of Solution Set for Example 3

WARNING

Students sometimes write

$$1 > x > 5$$

This is a misuse of the inequality notation since it states that x is simultaneously less than 1 *and* greater than 5, which is impossible. What is usually intended is the pair of infinite intervals $(-\infty, 1)$ and $(5, \infty)$, and the inequalities must be written

$$x < 1 \quad \text{or} \quad x > 5$$

Two additional misuses of the inequality notation are

$$1 < x > 5 \quad \text{and} \quad 1 > x < 5$$

We summarize some facts concerning absolute values in equations and inequalities.

If $a > 0$, then:

- $|x| = a$ is equivalent to $x = \pm a$.
- $|x| < a$ is equivalent to $-a < x < a$.
- $|x| > a$ is equivalent to $x < -a$ or $x > a$.

Verify these results using the Critical Value Method.

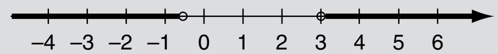
✔ Progress Check

Solve each inequality, write the solution set in interval notation, and graph the solution.

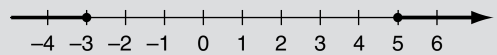
a. $|5x - 6| > 9$ b. $|2x - 2| \geq 8$

Answers

a. $\left(-\infty, -\frac{3}{5}\right), (3, \infty)$

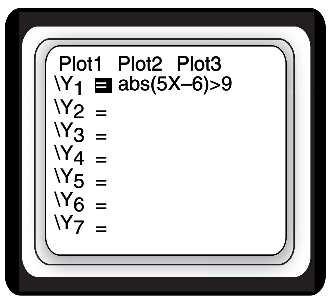


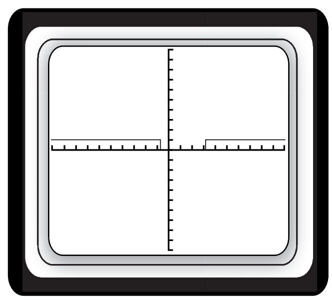
b. $(-\infty, -3], [5, \infty)$



EXAMPLE 4 GRAPHING THE SOLUTION TO AN ABSOLUTE VALUE INEQUALITY

You can use your graphing calculator to visualize the solution set of an absolute value inequality. Enter the inequality as seen below. Ignore the y-axis, and think of the x-axis as a number line. Note that the graph will not tell you whether the endpoints are included in the solution set.





Exercise Set 2.6

In Exercises 1–9, solve and check.

1. $|x + 2| = 3$
2. $|r - 5| = \frac{1}{2}$
3. $|2x - 4| = 2$
4. $|5y + 1| = 11$
5. $|-3x + 1| = 5$
6. $|2t + 2| = 0$
7. $3|-4x - 3| = 27$
8. $\frac{1}{|x|} = 5$
9. $\frac{1}{|s-1|} = \frac{1}{3}$

In Exercises 10–15, solve the inequality and graph the solution set.

10. $|x + 3| < 5$
11. $|x + 1| > 3$
12. $|3x + 6| \leq 12$
13. $|4x - 1| > 3$
14. $|3x + 2| \geq -1$
15. $\left|\frac{1}{3} - x\right| < \frac{2}{3}$

In Exercises 16–24, solve the inequality, and write the solution set using interval notation.

16. $|x-2| \le 4$

17. $|x-3| \ge 4$

18. $|2x+1| < 5$

19. $\dfrac{|2x-1|}{4} < 2$

20. $\dfrac{|3x+2|}{2} \le 4$

21. $\dfrac{|2x+1|}{3} < 0$

22. $\left|\dfrac{4}{3x-2}\right| < 1$

23. $\left|\dfrac{5-x}{3}\right| > 4$

24. $\left|\dfrac{2x+1}{3}\right| \le 5$

In Exercises 25–28, solve for x.

25. $|2x+1| - 3 = -2$

26. $3 - |2x+4| = 1$

27. $2|3-x| + 3 = 5$

28. $4 - 3|2x+7| = -5$

In Exercises 29 and 30, x and y are real numbers.

29. Prove that $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$ (*Hint:* Consider four cases.)

30. Prove that $|x|^2 = x^2$.

31. A machine that packages 100 vitamin pills per bottle can make an error of 2 pills per bottle. If x is the number of pills in a bottle, write an inequality, using absolute value, that indicates a maximum error of 2 pills per bottle. Solve the inequality.

32. The weekly income of a worker in a manufacturing plant differs from \$300 by no more than \$50. If x is the weekly income, write an inequality, using absolute value, that expresses this relationship. Solve the inequality.

33. Express the statement $x > 6$ or $x < -6$ as a single inequality using absolute value.

34. Express the statement $-10 < x < 10$ as an inequality using absolute value.

35. Express the statement $x \ge 5$ or $x \le 1$ as a single inequality using absolute value.

36. Express the statement $2d - 5 \le x \le 2d + 5$ as an inequality using absolute value.

37. Write an equation that states that x is 10 units from 4 on the real number line. Solve this equation.

38. Find all points x on the real number line such that x is 5 times as far from the origin as from 20.

39. Find all points x on the real number line such that x is 3 times as far from 4 as $2x$ is from 6.

40. Find the value of $\dfrac{|x| - x}{2}$ for

    a. $x \ge 0$

    b. $x < 0$

41. Chebyshev's Theorem from probability theory states that the probability that any random variable x will assume a value within k standard deviations σ of its mean μ is at least $1 - \frac{1}{k^2}$, or, equivalently,

$$P(|x-\mu| < k\sigma) \ge 1 - \frac{1}{k^2}$$

Solve $|x - \mu| < k\sigma$ for x.

42. In calculus, absolute value inequalities are used in the definitions of terms like *continuous function* and *limit*. Solve the following inequality for x when $\delta = 0.005$ and $x_0 =$ 0.001:

$$|x - x_0| < \delta$$

43. Solve the following inequality for x when $L = 9$ and ε has each of the following values: 0.01, 0.001, 0.0001.

$$|(2x+5) - L| < \varepsilon$$

■ ■ ■

Chapter Summary

Terms and Symbols

closed interval	146	half-open interval	132	repeated root	118
completing the square	113	identity	89	right-hand side (RHS)	88
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critical value	136	infinity, ∞	133	second-degree inequality	138
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double root	118	left-hand side (LHS)	88	solution of an inequality	130
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		radical equation	119		

Key Ideas for Review

Topic	Page	Key Idea
Solutions of an Equation	88	A solution of an equation is a value that satisfies the equation.
Solution Process	88	To solve an equation, we generally form a succession of simpler, equivalent equations. We may add to or subtract from both sides of the equation any number or expression. We may also multiply both sides by any nonzero number. If we multiply the equation by an expression containing a variable, the answers must be substituted into the original equation to verify that they are solutions.
Linear Equations	91	The linear equation $$ax + b = 0, \quad a \neq 0$$ has precisely one solution $$x = -\frac{b}{a}$$
Completing the Square	114	$$x^2 + dx + \frac{d^2}{4} = \left(x + \frac{d}{2}\right)^2$$ Therefore, add $\frac{d^2}{4}$ to $x^2 + dx$ to "complete the square."
Quadratic Equations	107	The quadratic equation $$ax^2 + bx + c = 0, \quad a \neq 0$$

Key Ideas for Review

Topic	Page	Key Idea
		always has two solutions that are given by the quadratic formula $$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}, \quad a \neq 0$$ If $b = 0$, or if the quadratic equation can be factored, then faster solution methods are available.
Discriminant	117	The solutions or roots of a quadratic equation may be complex numbers. The expression $$b^2 - 4ac$$ under the radical of the quadratic formula is called the discriminant. Its value determines the nature of the roots of the quadratic equation.
Radical Equations	119	Radical equations often can be transformed into quadratic equations. Since the process involves raising both sides of an equation to a power, the answers must be checked to see that they satisfy the original equation.
Substitution of Variable	122	The method called *substitution of variable* can be used to transform certain equations into quadratic equations. This is a valuable technique that will be used in other chapters of this book.
Solutions of an Inequality	130	A solution of an inequality is a value that satisfies the inequality.
Solution Process	131	The permissible operations in solving an inequality are the same as those for solving equations with this proviso: multiplication or division by a negative number reverses the direction of the inequality.
Solution Set Representation	133	The solution set of an inequality can be represented by using set notation, interval notation, or a graph on the real number line.
Critical Value Method	136	The critical values of an inequality are as follows: 1. those values for which either side of the inequality is not defined (such as a denominator equal to 0) 2. those values that are solutions to the equation obtained by replacing the inequality sign with an equal sign An inequality can be solved by finding the critical values and checking a test point in each interval determined by those critical values.

Review Exercises

Solutions to exercises whose numbers are in blue are in the Solutions section in the back of the book.

In Exercises 1–4, solve for x.

1. $3x - 5 = 3$
2. $2(2x - 3) - 3(x + 1) = -9$
3. $\dfrac{2 - x}{3 - x} = 4$
4. $k - 2x = 4kx$
5. The width of a rectangle is 4 centimeters less than twice its length. If the perimeter is 12 centimeters, find the measurement of each side.
6. A donation box contains coins consisting of dimes and quarters. The number of dimes is 4 more than twice the number of quarters. If the total value of the coins is \$2.65, how many coins of each type are there?
7. It takes 4 hours for a bush pilot in Australia to pick up mail at a remote village and return to home base. If the average speed going is 150 mph and the average speed returning is 100 mph, how far from the home base is the village?
8. Copying machines A and B, working together, can prepare enough copies of the annual report for the board of directors in 2 hours. Machine A, working alone, would require 3 hours to do the job. How long would it take machine B to do the job by itself?

In Exercises 9 and 10 indicate whether the statement is true (T) or false (F).

9. The equation $3x^2 = 9$ is an identity.
10. $x = 3$ is a solution of the equation $3x - 1 = 10$.
11. Solve $x^2 - x - 20 = 0$ by factoring.
12. Solve $6x^2 - 11x + 4 = 0$ by factoring.
13. Solve $x^2 - 2x + 6 = 0$ by completing the square.
14. Solve $2x^2 - 4x + 3 = 0$ by the quadratic formula.
15. Solve $3x^2 + 2x - 1 = 0$ by the quadratic formula.

In Exercises 16–18, solve for x.

16. $49x^2 - 9 = 0$
17. $kx^2 - 3\pi = 0$
18. $x^2 + x = 12$

In Exercises 19–21, determine the nature of the roots of the quadratic equation without solving.

19. $3r^2 = 2r + 5$
20. $4x^2 + 20x + 25 = 0$
21. $6y^2 - 2y = -7$

In Exercises 22–25, solve the given equation.

22. $\sqrt{x} + 2 = x$
23. $\sqrt{x + 3} + \sqrt{2x - 3} = 6$
24. $x^4 - 4x^2 + 3 = 0$
25. $\left(1 - \dfrac{2}{x}\right)^2 - 8\left(1 - \dfrac{2}{x}\right) + 15 = 0$
26. A charitable organization rented an auditorium for a meeting at a cost of \$420 and split the cost among the attendees. If 10 additional persons had attended the meeting, the cost per person would have decreased by \$1. How many persons actually attended?
27. Solve and graph $3 \leq 2x + 1$.
28. Solve and graph $-4 < -2x + 1 \leq 10$.

In Exercises 29–31, solve the inequality and express the solution set in interval notation.

29. $2(a + 5) > 3a + 2$
30. $\dfrac{-1}{2x - 5} \leq 0$

Review Exercises

31. $\frac{2x}{3} + \frac{1}{2} \geq \frac{x}{2} - 1$

32. Solve $|3x + 2| = 7$ for x.

33. Solve and graph $|4x - 1| = 5$.

34. Solve and graph $|2x + 1| > 7$.

35. Solve $|2 - 5x| < 1$ and write the solution in interval notation.

36. Solve $|3x - 2| \geq 6$ and write the solution in interval notation.

37. Find the values of x for which $\sqrt{2x^2 - x - 6}$ has real values.

38. Using interval notation, write the solution set of the inequality $x^2 + 4x - 5 \leq 0$.

39. Write the solution set for $\frac{2x + 1}{x + 5} \geq 0$ in interval notation.

40. Write the solution set for

$$(3 - x)(2x + 3)(x + 2) < 0$$

in interval notation.

41. A local school board is debating the question of whether or not to close an elementary school in its district. The board expects a larger than average number of local residents at this meeting. The typical meeting seats the residents in a rectangular formation of 10 rows, 15 seats to a row. In order to double the seating capacity with a new rectangular formation, the board decides to add an equal number of seats to each existing row and to add that same number of additional rows to the original formation. Find the number of chairs needed to add to each row.

42. A circle and a square have the same perimeter P. Show that the area of the circle is greater than the area of the square.

43. A rectangular window is to be cut into a door 6.5 feet tall so that the distance below the window is 6 inches less than twice the distance above the window. The window must be 18 inches wide and have an area of at least 324 square inches. What is the maximum distance between the top of the door and the top of the window?

44. In statistical theory, one can develop an internal estimate of the population mean μ with a known standard deviation σ by taking a sample of size n and finding the mean $\bar{x}$ of the sample. The 95% confidence interval is given by

$$\left| \frac{\mu - \bar{x}}{\frac{\sigma}{\sqrt{n}}} \right| < 1.96$$

Solve this inequality for μ.

45. The kinetic energy E of a particle depends on the particle's mass m and speed v and is defined as

$$E = \frac{1}{2}mv^2$$

a. A 10,000-kilogram bullet has kinetic energy of 5000 kilogram meters per second squared. Find the speed of the bullet.

b. If the mass of the bullet is halved, find the speed.

46. An open box is to be constructed from a piece of cardboard that is x inches by $x + 2$ inches. This is accomplished by cutting 2-inch squares from each corner of the cardboard and bending up the sides. If the volume of the box is to be 96 cubic inches, find the dimensions of the box.

Review Test

In Problems 1 and 2, solve for y.

1. $5 - 4y = 2$
2. $\dfrac{2 + 5y}{3y - 1} = 6$
3. One side of a triangle is 2 meters shorter than the base, and the other side is 3 meters longer than half the base. If the perimeter is 15 meters, find the length of each side.
4. A trust fund invested a certain amount of money at 6.5% simple annual interest, a second amount \$200 more than the first amount at 7.5%, and a third amount \$300 more than twice the first amount at 9%. If the total annual income from these investments is \$1962, how much was invested at each rate?
5. Indicate whether the statement is true (T) or false (F): The equation $(2x - 1)^2 = 4x^2 - 4x + 1$ is an identity.
6. Solve $x^2 - 5x = 14$ by factoring.
7. Solve $5x^2 - x + 4 = 0$ by completing the square.
8. Solve $12x^2 + 5x - 3 = 0$ by the quadratic formula.

In Problems 9 and 10, solve for x.

9. $(2x - 5)^2 + 9 = 0$
10. $2 + \dfrac{1}{x} - \dfrac{3}{x^2} = 0$

In Problems 11 and 12, determine the nature of the roots of the quadratic equation without solving.

11. $6x^2 + x - 2 = 0$
12. $3x^2 - 2x = -6$

In Problems 13 and 14, solve the given equation.

13. $x - \sqrt{4 - 3x} = -8$
14. $3x^4 + 5x^2 - 2 = 0$
15. The area of a rectangle is 96 square meters. If the length and the width are each increased by 2 meters, the area of the newly formed rectangle is 140 square meters. Find the dimensions of the original rectangle.
16. Solve $-1 \le 2x + 3 < 5$ and graph the solution set.

In Problems 17 and 18, solve the inequality and express the solution set in interval notation.

17. $3(2a - 1) - 4(a + 2) \le 4$
18. $-2 \le 2 - x \le 6$
19. Solve $|4x - 1| = 9$.
20. Solve $|2x - 1| \le 5$ and graph the solution set.
21. Solve $|1 - 3x| > 5$ and write the solution in interval notation.
22. Find the values of x for which $\sqrt{3x^2 - 4x + 1}$ has real values.

In Problems 23–25, write the solution set in interval notation.

23. $-2x^2 + 3x - 1 \le 0$
24. $(x - 1)(2 - 3x)(x + 2) \le 0$
25. $\dfrac{2x - 5}{x + 1} > -\dfrac{1}{3}$

Writing Exercises

In Exercises 1–3, write in complete sentences the procedure that you follow in solving the following problems.

1. An automatic teller machine gives you \$150 in five and ten dollar bills. There are 2 more than twice as many five dollar bills as there are ten dollar bills. How many of each denomination are there?
2. A pound of raisins costs \$2.50 whereas a pound of chocolate bits costs \$4.00. If we want to make a one-pound mixture of these items to sell for \$3.25, how much of each item must be used?
3. Two students work in a library shelving books. Eric can shelve 50 books per hour and Steve can shelve 40 books per hour. If Steve starts work in the morning and then is relieved by Eric later in the day, how long did each student work if 355 books were shelved in an 8-hour day?
4. Why is it a good practice to check your answers? Give an example to show how not following this practice can lead to a wrong conclusion.

Chapter 2 Project

Have you ever considered the challenges involved in linking computers and databases across the world into a user-friendly network? The mathematics involved in any such project are very sophisticated. The type of network we discussed in this chapter is far simpler!

In Section 2.3, do Exercise 119, and in Section 2.5, do Exercise 116.

In a **central relay network**, all users are connected to one central point. Therefore, the number of users is exactly equal to the number of links necessary. For how many users would a central relay network actually require fewer links than a full-mesh network? The same number? Sketch a diagram to illustrate.