7 The Trigonometric Functions

The word "trigonometry" derives from the Greek, meaning "measurement of triangles." Historical interest in trigonometry was strongly motivated by the development of such fields as navigation, astronomy, and surveying. Hipparchus of Nicaea (190–125 B.C.) is often referred to as the "father of trigonometry."

We begin this chapter with the classical approach to trigonometry, that is, examining the relationships between the sides and angles of right triangles. We then follow with the more modern approach that emphasizes the concept of a function, as introduced in Chapter 3. Here, the trigonometric functions are viewed as functions of real numbers whose ranges are related to points on a circle. Thus, they are also called *circular functions.*

After examining the graphs of the trigonometric functions, we consider the possibility of inverses of these functions. We conclude the chapter with some applications.

7.1 Angles and Their Measurement

Definition of an Angle

In plane geometry, an angle is usually formed by the sides of a triangle. In trigonometry, we need to introduce a more general concept of an angle. Here, an angle is unlimited in magnitude, and it can be either positive or negative.

An **angle** is the geometric shape formed by two rays, or half-lines, with a common endpoint. For our purposes, we wish to define an angle as the result of a rotation of a ray about its endpoint. In Figure 1, the **initial side** is rotated about its endpoint O until it coincides with the **terminal side** to form the angle α. When the ray coincides with the positive x-axis with its endpoint at the origin, the angle generated is said to be in **standard position.** In Figure 2(a), the x-axis, called the **initial side,** rotates in a counterclockwise direction until it coincides with the terminal side, forming the angle α. In this case we say that α is a **positive angle.** In Figure 2(b), the ray has been rotated in a clockwise direction to form the angle β. In this case, we say that β is a **negative angle.** If the terminal side coincides with a coordinate axis, the angle is called a **quadrantal angle.** In all other cases, the angle is said to lie in the same quadrant as its terminal side. Figure 3 displays an angle in each of the four quadrants. Note that the designation depends only upon the quadrant in which the terminal side lies and not upon the direction of rotation.

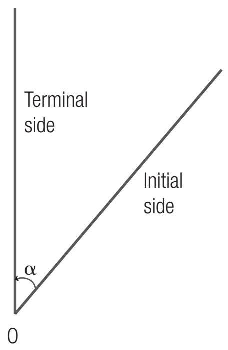


FIGURE 1 Forming an Angle

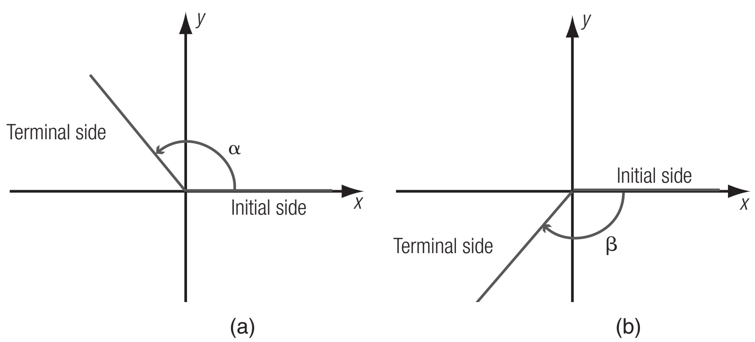


FIGURE 2 Positive and Negative Angles in Standard Position

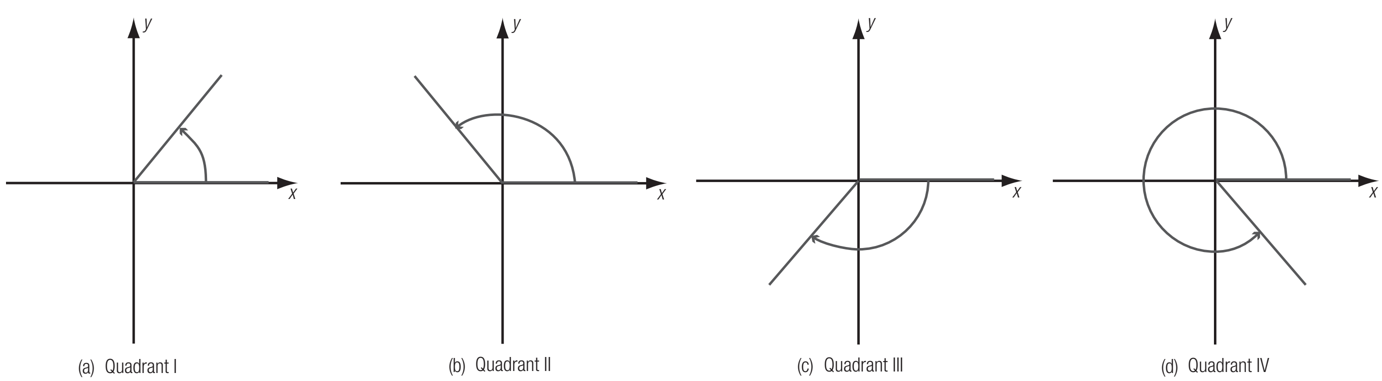


FIGURE 3 An Angle in Each Quadrant

Angular Measurement: Degrees and Radians

There are two commonly used units for measuring angles. An angle is said to have a measure of one **degree**, written 1°, if the angle is formed by rotating the initial side $\frac{1}{360}$ of a complete rotation in a counterclockwise direction. It follows that an angle obtained by a complete rotation of the initial side has a measure of 360°, and an angle obtained by one-fourth of a complete rotation has a measure of $\frac{1}{4}(360°) = 90°$. One degree is subdivided into 60 **minutes,** written 60′, and one minute is subdivided into 60 **seconds,** written 60″. For example, the notation 14°24′18″ is read *14 degrees, 24 minutes and 18 seconds.* This is equivalent to 14.405°, since

$$\begin{aligned} 14°24'18'' &= 14°24' + \left(\frac{18}{60}\right)' = 14°24' + 0.3' \\ &= 14°24.3' = 14° + \left(\frac{24.3}{60}\right)° \\ &= 14° + 0.405° = 14.405° \end{aligned}$$

An angle in standard position greater than 0° and less than 90° lies in the first quadrant and is called an **acute angle**. (See Figure 3(a).) An angle greater than 90° and less than 180° lies in the second quadrant and is called an **obtuse angle**. (See Figure 3(b).) An angle measuring 90° is a quadrantal angle and is called a **right angle**. Angles measuring 0°, 180°, 270°, 360°, and −90° are also examples of quadrantal angles.

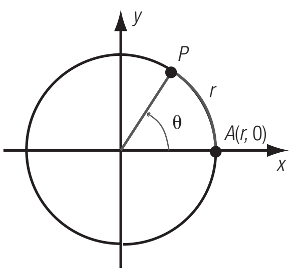


FIGURE 4 An Angle of 1 Radian

To define the second unit of angular measurement, consider a circle centered at the origin with radius r, as shown in Figure 4. Choose point P on the circle, counterclockwise from A, so that the arc $\widehat{AP}$ has length equal to r. Point P thus determines the angle θ. As the length or measure of the arc $\widehat{AP}$ is *1 radius,* we say that the measure of the angle θ is one **radian.** From this definition, we can find the radian measure of any angle. For example, in Figure 5(a), the length of arc $\widehat{AP}$ is $2r$. Therefore, θ is an angle of 2 radians. In Figure 5(b), the length of arc $\widehat{AP}$ is $\frac{1}{2}r$. In this case, θ is an angle of $\frac{1}{2}$ radian. Figure 5(c) shows the arc $\widehat{AP}$ of length $\frac{2}{3}r$ in the clockwise direction. Here, θ is an angle of $-\frac{2}{3}$ radian ≈ −0.6667 radians. In Figure 6(a), the length of the arc $\widehat{AP}$ is the length of the circumference of the circle, the formula for which is $2\pi r \approx 6.2832r$. Therefore, $\theta = 2\pi$ radians ≈ 6.2832 radians. In Figure 6(b), the the length of the arc $\widehat{AP}$ is half the circumference of the circle, namely, $\pi r \approx 3.1416r$. In this case, $\theta = \pi$ radians ≈ 3.1416 radians.

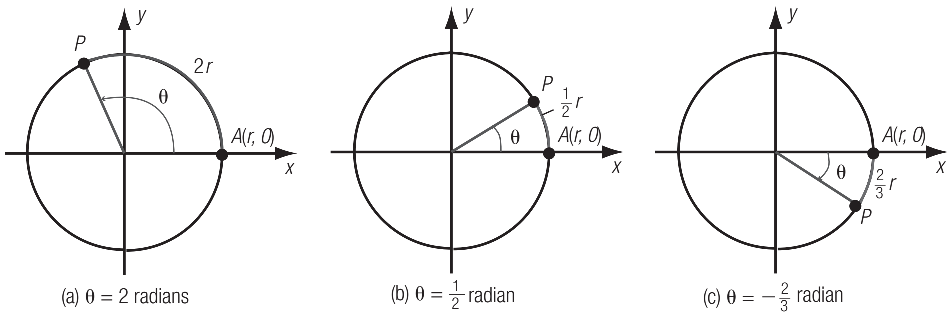


FIGURE 5 Measuring Angles in Radians

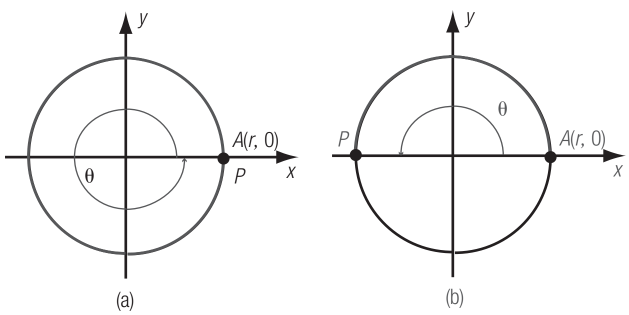


FIGURE 6 Measuring Angles in Radians

Angle Conversion

An angle of 360° traces a complete revolution in the counterclockwise direction. As we have just observed in the discussion concerning Figure 6(a), that same angle has a measure of 2π radians. Therefore, 2π radians $= 360°$ or

$$\pi \text{ radians} = 180° \tag{1}$$

This relationship enables us to transform angular measure from radians to degrees and vice versa. One way to handle such conversions for any angle θ is by establishing a proportion.

$$\frac{\text{radian measure of angle } \theta}{\pi \text{ radians}} = \frac{\text{degree measure of angle } \theta}{180°}$$

Alternatively, we can solve Equation (1) to provide the conversion formulas

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^{\circ} \qquad \text{and} \qquad 1° = \frac{\pi}{180} \text{ radians}$$

Equivalently,

$$1 \text{ radian} \approx 57.29578° \qquad \text{and} \qquad 1° \approx 0.0174533 \text{ radians}$$

EXAMPLE 1 ANGLE CONVERSION

Convert 150° to radian measure.

SOLUTION

With θ representing the radian measure of the angle, we establish the proportion

$$\frac{\theta}{\pi} = \frac{150^\circ}{180^\circ}$$

Solving, we have

$$\theta = \frac{150\pi}{180} = \frac{5\pi}{6}$$

Alternatively, since $1^\circ = \frac{\pi}{180}$ radians, we have

$$150^\circ = 150\left(\frac{\pi}{180}\right) = \frac{5\pi}{6} \text{ radians}$$

Thus, $\theta = \frac{5\pi}{6}$ radians.

✔ Progress Check

Convert the following from degree to radian measure.

(a) -210° (b) 390°

Answers

(a) $-\dfrac{7\pi}{6}$ radians (b) $\dfrac{13\pi}{6}$ radians

EXAMPLE 2 ANGLE CONVERSION

Convert $\frac{2\pi}{3}$ radians to degree measure.

SOLUTION

With θ denoting the degree measure of the angle, we establish the proportion

$$\frac{\frac{2\pi}{3}}{\pi} = \frac{\theta}{180^\circ}$$

Solving, we have

$$\theta = \frac{2}{3}(180^\circ) = 120^\circ$$

Alternatively, since 1 radian $= \frac{180}{\pi}$ degrees,we have

$$\frac{2\pi}{3} \text{ radians} = \frac{2\pi}{3}\left(\frac{180}{\pi}\right)^\circ = 120^\circ$$

Thus, $\theta = 120^\circ$.

✔ Progress Check

Convert the following from radian measure to degrees.

(a) $\dfrac{9\pi}{2}$ radians (b) $-\dfrac{4\pi}{3}$ radians

Answers

(a) $810°$ (b) $-240°$

Graphing Calculator Power User's Corner 7.1

Converting Angle Measure

To determine whether your calculator is presently interpreting angles as radians or degrees, press the mode button. If it is not in radian mode, change it now. Most people prefer to leave the graphing calculator in radian mode when doing trigonometry. Then if you are in need of a degree measurement, you can easily convert with the calculator. If you want to enter an angle that is in degree measure such as 39°, you simply type 39[2nd][APPS] and choose the degree symbol, #1. You can use your calculator to convert angles from radians to degrees, minutes, and seconds in the following way:

Instruction	Screen View
Type in the radian value of the angle 8π by entering [2nd][^].	8π
Type [2nd][APPS] for the angle menu. Choose #4, >DMS, which is the command to convert to degrees, minutes, and seconds.	8π > DMS
Press enter. The result is displayed in degrees, minutes, and seconds, accurate to 1/1000 of a second.	25°7′57.868″

Practice Problems

Convert the following into degree measure:

1. $\pi/3$
2. 6.283
3. $7\pi/6$
4. $-\pi/4$

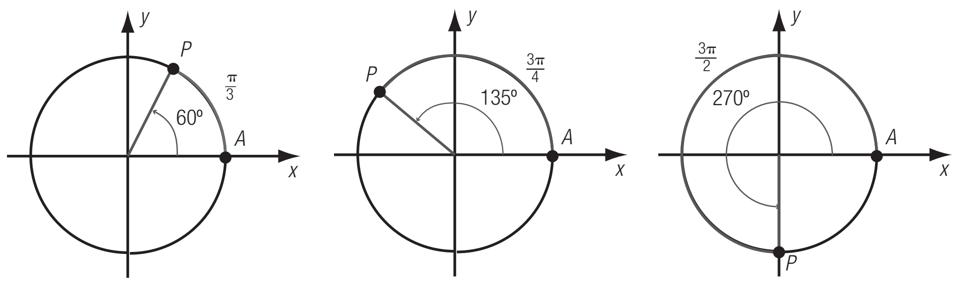
"Special" Angles

There are certain angles that we will use frequently in the examples and exercises throughout this chapter. It may prove helpful to verify the conversions shown in Table 1. Figure 7 displays some angles in standard position and shows both the radian and degree measure.

TABLE 1 Radians and Degrees

Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
Degrees	0°	30°	45°	60°	90°	120°	135°	150°	180°

Radians	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
Degrees	210°	225°	240°	270°	300°	315°	330°	360°



(a) $\theta = 60° = \frac{\pi}{3}$ radians (b) $\theta = 135° = \frac{3\pi}{4}$ radians (c) $q = 270° = \frac{3\pi}{2}$ radians

FIGURE 7 Degree and Radian Measure

EXAMPLE 3 FINDING QUADRANTS

If the angle θ is in standard position, determine the quadrant in which the given angle lies.

(a) $\theta = 200°$ (b) $\theta = \dfrac{7\pi}{4}$ radians

SOLUTION

Figure 8 shows the quadrantal angles in standard position.

(a) Since $\theta = 200°$ is between 180° and 270°, θ lies in quadrant III.

(b) Since $\theta = \frac{7\pi}{4}$ radians is between $\frac{3\pi}{2}$ and 2π, θ lies in quadrant IV.

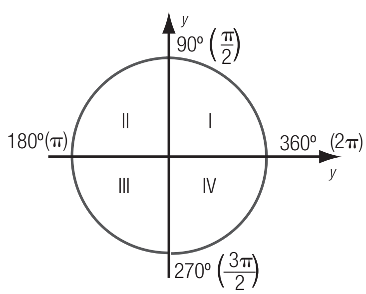


FIGURE 8 Quadrants and Angles

The Reference Angle

Since a complete revolution about a circle returns to the initial position, different angles in standard position may have the same terminal side. For example, angles of 30° and 390° in standard position have the same terminal side, as shown in Figure 9(a). Similarly, the angles of 45° and −315° shown in Figure 9(b) have the same terminal side. Such angles are said to be **coterminal**.

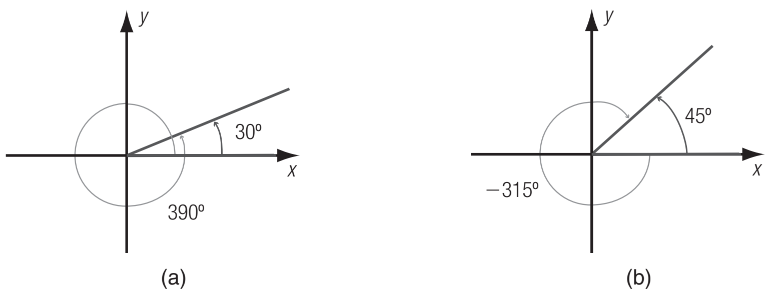


FIGURE 9 Coterminal Angles

Any number of complete revolutions in either direction will return to the same terminal side. Since a revolution requires 360°, or 2π radians, we obtain the following result.

Coterminal Angles

An angle θ in standard position is coterminal with every angle of the form

$$\theta + 360° n \qquad \text{(degree measure)}$$

or

$$\theta + 2\pi n \qquad \text{(radian measure)}$$

where n is an integer.

EXAMPLE 4 FINDING COTERMINAL ANGLES

Find a first quadrant angle that is coterminal with an angle of

(a) 410° (b) $-\frac{5\pi}{3}$ radians

SOLUTION

(a) Since $410° - 360° = 50°$, it follows that an angle of 50° is coterminal with an angle of 410°.

(b) Since

$$-\frac{5\pi}{3} + 2\pi = \frac{\pi}{3}$$

$\frac{\pi}{3}$ and $-\frac{5\pi}{3}$ are coterminal.

✔ Progress Check

Show that each pair of angles is coterminal.

(a) $-265°$ and $95°$ (b) $\dfrac{22\pi}{3}$ and $\dfrac{4\pi}{3}$ radians

For an angle in standard position that is not a quadrantal angle, it is convenient to define an acute angle that is called the **reference angle**.

The reference angle θ' associated with the angle θ is the acute angle formed by the terminal side of θ and the *x*-axis.

If θ lies in quadrant I and it is an acute angle, then $\theta' = \theta$. Other cases are illustrated in Figure 10. Therefore, when the reference angle is measured in radians, it is in the interval $(0, \frac{\pi}{2})$. When the reference angle is measured in degrees, it is in the interval $(0°, 90°)$.

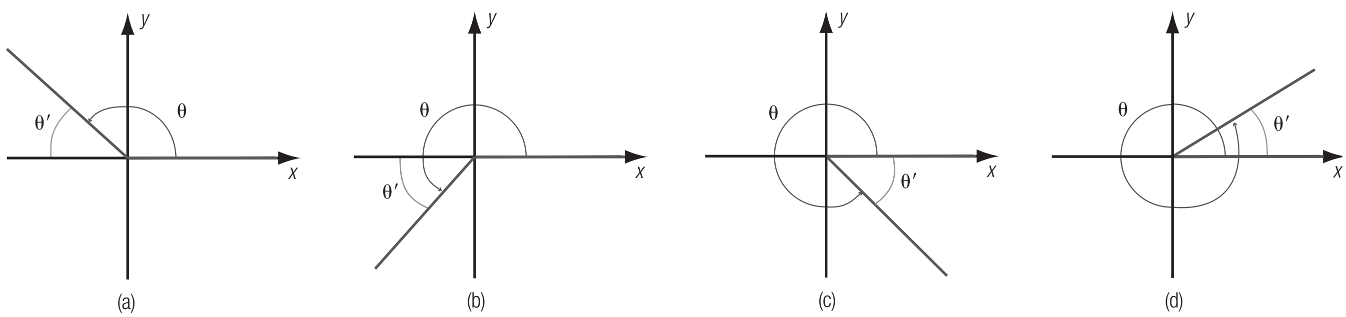


FIGURE 10 The Reference Angle

EXAMPLE 5 REFERENCE ANGLE COMPUTATION

Find the reference angle θ' if

(a) $\theta = 240°$ (b) $\theta = \dfrac{5\pi}{3}$ radians

SOLUTION

(a) Since $\theta = 240°$ lies in the third quadrant, the reference angle is

$$\theta' = 240° - 180° = 60°$$

(b) Since $\theta = \frac{5\pi}{3}$ radians lies in the fourth quadrant, the reference angle is

$$\theta' = 2\pi - \frac{5\pi}{3} = \frac{\pi}{3}$$

✔ Progress Check

Find the reference angle θ' if

(a) $\theta = 160°$ (b) $\theta = \frac{4\pi}{3}$ radians

Answers

(a) $20°$ (b) $\frac{\pi}{3}$ radians

Length of a Circular Arc and the Central Angle

Consider two concentric circles with radii r_1 and r_2 as shown in Figure 11. We let s_1 and s_2 be the lengths of arcs $\widehat{A_1P_1}$ and $\widehat{A_2P_2}$, respectively. From plane geometry, we know that

$$\frac{r_1}{r_2} = \frac{s_1}{s_2}$$

Therefore, we may rewrite this proportion as

$$\frac{s_1}{r_1} = \frac{s_2}{r_2}$$

The ratio $\frac{s_1}{r_1}$ is precisely the radian measure of the angle 0, as is $\frac{s_2}{r_2}$. Equivalently, we have

Length of a Circular Arc

If a central angle θ subtends an arc of length s on a circle of radius r, then the radian measure of θ is given by

$$\theta = \frac{s}{r}$$

and the length of the arc s is given by

$$s = r\theta$$

Note that if the length of the arc s is equal to the length of one radius r, then the measure of θ is one radian.

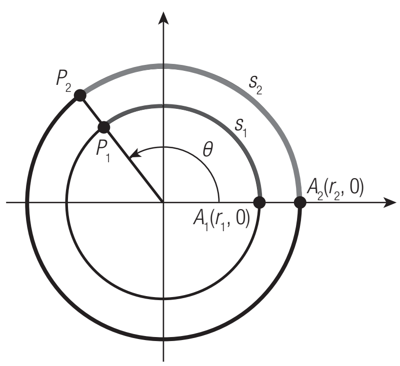


FIGURE 11 Radian Measure with Any Circle

EXAMPLE 6 THE CENTRAL ANGLE FORMULA

A central angle θ subtends an arc of length 12 inches on a circle whose radius is 6 inches. Find the radian measure of the central angle.

SOLUTION

We have $s = 12$ and $r = 6$, so that

$$\theta = \frac{s}{r} = \frac{12}{6} = 2 \text{ radians}$$

WARNING

The formula

$$\theta = \frac{s}{r}$$

can only be applied if the angle θ is in radian measure.

EXAMPLE 7 THE CENTRAL ANGLE FORMULA

A designer has to place the word "ALMONDS" on a can using equally spaced letters, as shown in Figure 12(a). For good visibility, the letters must cover a sector of the circle having a 90° central angle. If the base of the can is a circle of radius 2 inches, as shown in Figure 12(b), what is the maximum width of each letter, ignoring the spacing between letters?

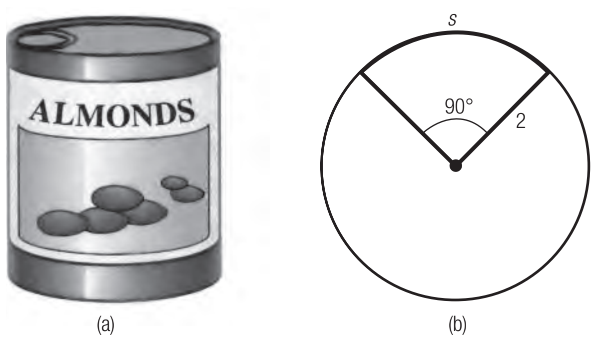


FIGURE 12 Diagram for Example 7

SOLUTION

Since $\theta = 90° = \frac{\pi}{2}$ radians, the arc has length

$$s = r\theta = 2\left(\frac{\pi}{2}\right) = \pi$$

Each of the seven letters can occupy $\frac{1}{7}$ of this arc, namely, $\frac{\pi}{7}$ inches.

Exercise Set 7.1

In Exercises 1–18, the angle θ is in standard position. Determine the quadrant in which the angle lies.

1. $\theta = 313°$
2. $\theta = 182°$
3. $\theta = 14°$
4. $\theta = 227°$
5. $\theta = 141°$
6. $\theta = -167°$
7. $\theta = -345°$
8. $\theta = 555°$
9. $\theta = 618°$
10. $\theta = -428°$
11. $\theta = -195°$
12. $\theta = 730°$
13. $\theta = \dfrac{7\pi}{8}$
14. $\theta = -\dfrac{3\pi}{5}$
15. $\theta = -\dfrac{8\pi}{3}$
16. $\theta = \dfrac{3\pi}{8}$
17. $\theta = \dfrac{13\pi}{3}$
18. $\theta = \dfrac{9\pi}{5}$

In Exercises 19–34, convert from degree measure to radian measure. Do not use a calculator in Exercises 19–30.

19. $30°$
20. $200°$
21. $-150°$
22. $-330°$
23. $75°$
24. $570°$
25. $-450°$
26. $-570°$
27. $135°$
28. $405°$
29. $120°$
30. $90°$
31. $45.22°$
32. $196.54°$
33. $123°20'$
34. $87°12'30''$

In Exercises 35–48, convert from radian measure to degree measure. Do not use a calculator in Exercises 35–46.

35. $\dfrac{\pi}{4}$
36. $\dfrac{\pi}{3}$
37. $\dfrac{3\pi}{2}$
38. $\dfrac{5\pi}{6}$
39. $-\dfrac{\pi}{2}$
40. $-\dfrac{7\pi}{12}$
41. $\dfrac{4\pi}{3}$
42. 3π
43. $\dfrac{5\pi}{2}$
44. -5π
45. $-\dfrac{5\pi}{3}$
46. $\dfrac{9\pi}{2}$
47. 1.72
48. 24.98

In Exercises 49–54, for each pair of angles, write T if they are coterminal and F if they are not coterminal.

49. $30°, 390°$
50. $50°, -310°$
51. $45°, -45°$
52. $120°, \dfrac{14\pi}{3}$
53. $\dfrac{\pi}{2}, \dfrac{7\pi}{2}$
54. $-60°, 760°$

In Exercises 55–66, for each given angle, find the reference angle.

55. $130°$
56. $\dfrac{5\pi}{6}$
57. $-20°$
58. $25°$
59. $-455°$
60. $700°$
61. $\dfrac{12\pi}{5}$
62. $\dfrac{5\pi}{4}$
63. $72°$
64. $-\dfrac{2\pi}{3}$
65. $\dfrac{9\pi}{4}$
66. $\dfrac{5\pi}{3}$

67. If a central angle θ subtends an arc of length 4 centimeters on a circle of radius 7 centimeters, find the approximate measure of θ in radians and in degrees.

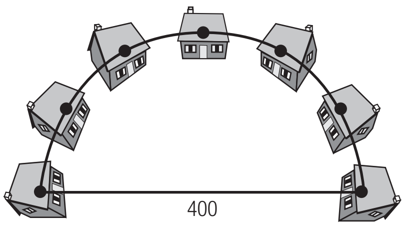
68. Find the length of an arc subtended by a central angle of $\frac{\pi}{5}$ radians on a circle of radius 6 inches.

69. Find the radius of a circle if a central angle of $\frac{2\pi}{3}$ radians subtends an arc of 4 meters.

70. In a circle of radius 150 centimeters, what is the length of an arc subtended by a central angle of $45°$?

71. A subcompact car uses a tire whose radius is 13 inches. How far has the car moved when the tire completes one rotation? How many rotations are completed when the tire has traveled 1 mile?

72. A builder intends placing 7 equally spaced homes on a semicircular plot as shown in the accompanying figure. If the circle has a diameter of 400 feet, what is the distance between any two adjacent homes?

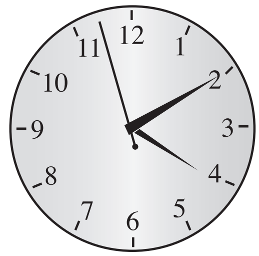


73. How many ribs are there in an umbrella if the length of each rib is 1.5 feet and the arc between two adjacent ribs measures $\frac{3\pi}{10}$ feet?

74. A microcomputer has both $5\frac{1}{4}$-inch and $3\frac{1}{2}$-inch floppy disk drives. If the disks used on both drives are divided into 8 sectors, find the ratio of the arc length of a sector of the larger disk to a sector of the smaller disk.

75. In a circle of radius r, a central angle of d degrees subtends an arc of length s. Establish a formula for s in terms of the variables r and d.

76. The minute hand on a clock is 3 inches long. Through what distance does the tip of the minute hand move between the hours of 4:10 P.M. and 4:22 P.M.?



77. A steam roller, whose wheel has a diameter of 2 feet, has just completed a pass over a driveway that is 30 feet in length. How many rotations of the roller were required?

78. The diameter of the earth is approximately 7900 miles. What is the difference in latitude between Lexington, Kentucky, and Jacksonville, Florida, if one city is 500 miles due north of the other? (*Hint:* Latitude is expressed in degrees.)

7.2 Right Triangle Trigonometry

Historically, trigonometry developed as a study of triangles. We begin by investigating various properties of right triangles. In Figure 13, we show right triangle ABC.

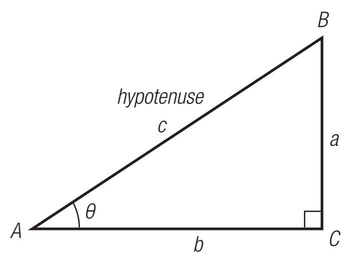


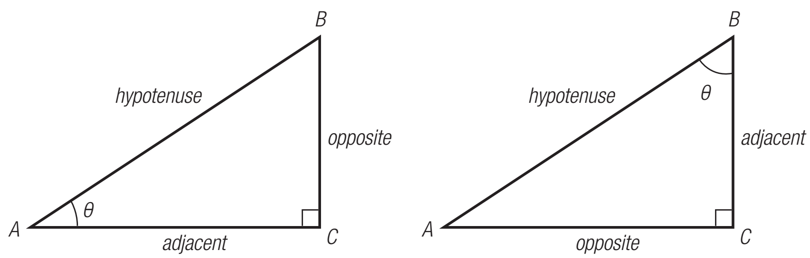
FIGURE 13 Right Triangle ABC

The letters A, B, and C play a dual role. They identify the vertices of this triangle, and they also identify the angles formed by the sides of the triangle at the respective vertices. Furthermore, in Figure 13, θ and A refer to the same angle.

From plane geometry, we know that the sum of the angles of a triangle equals 180°. Since angle C is a right angle (90°), we have that

$$\text{angle } A + \text{angle } B = 90^\circ$$

From this we conclude that both A and B are acute angles. The length of the side opposite the right angle C is called the **hypotenuse.** It will be called $\overline{\text{AB}}$, c, or *hypotenuse*. With our focus on angle θ, we identify the two legs or other two sides of triangle ABC. The length of the side opposite θ will be called $\overline{\text{BC}}$, a, or *opposite*. The length of the side adjacent to θ will be called $\overline{\text{AC}}$, b, or *adjacent*. The terms "opposite" and "adjacent" are names that may only be applied when our focus is directed toward a particular angle. In Figure 14(a), we focus on angle A and note the labels on the legs of triangle ABC as described above. In Figure 14(b), our focus is on angle B and the labels on the legs are reversed from those of Figure 14(a).

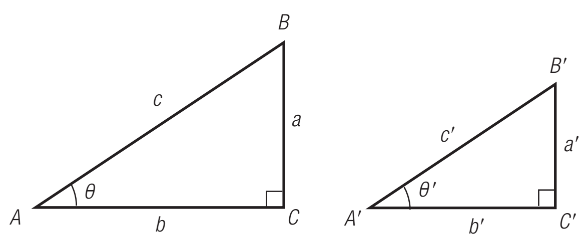


(a) Focus on Angle A (b) Focus on Angle B

FIGURE 14 Right Triangle ABC

Now consider the two right triangles shown in Figure 15. If $\theta = \theta'$, then these two triangles must be similar. (Why?) The similarity of these triangles implies that their corresponding sides are proportional, that is,

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$



(a) (b)

FIGURE 15 Similar Right Triangles ABC and $A'B'C'$

Alternatively, we may write

$$\frac{a}{b} = \frac{a'}{b'}, \qquad \frac{a}{c} = \frac{a'}{c'}, \qquad \frac{b}{c} = \frac{b'}{c'}$$

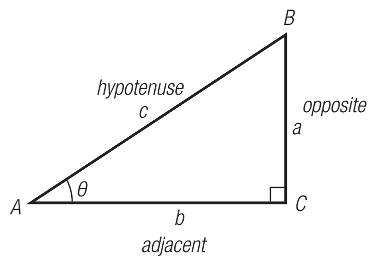
This indicates that the ratios of the corresponding sides are equal, hence, *independent* of the "size" of the right triangle. Because of this independence, consider all possible ratios of two sides of triangle ABC:

$$\frac{a}{c}, \quad \frac{b}{c}, \quad \frac{a}{b}, \quad \frac{b}{a}, \quad \frac{c}{b}, \quad \frac{c}{a}$$

These six ratios are used to define the six **trigonometric functions: sine, cosine, tangent, cotangent, secant**, and **cosecant**. We present the definitions of these functions in Table 2, which also contains triangle ABC.

Definition of sin θ, cos θ, tan θ, cot θ, sec θ, csc θ

TABLE 2 Definition of Trigonometric Functions



Function Name	Notation and Definition
sine	$\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}} = \dfrac{a}{c}$
cosine	$\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{b}{c}$
tangent	$\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}} = \dfrac{a}{b}$
cotangent	$\cot\theta = \dfrac{\text{adjacent}}{\text{opposite}} = \dfrac{b}{a}$
secant	$\sec\theta = \dfrac{\text{hypotenuse}}{\text{adjacent}} = \dfrac{c}{b}$
cosecant	$\csc\theta = \dfrac{\text{hypotenuse}}{\text{opposite}} = \dfrac{c}{a}$

Before we consider some examples, we wish to recall the Pythagorean Theorem as applied to the right triangle in Table 2.

$$a^2 + b^2 = c^2$$

or equivalently,

$$(\textit{opposite})^2 + (\textit{adjacent})^2 = (\textit{hypotenuse})^2$$

Whereas this formula may be helpful in solving problems, it will also be useful in finding important relationships among the trigonometric functions.

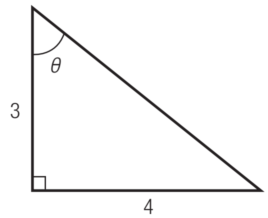


FIGURE 16 Diagram for Example 1

EXAMPLE 1 RIGHT TRIANGLE TRIGONOMETRY

Find the values of the trigonometric functions of the angle θ in Figure 16.

SOLUTION

Using the Pythagorean Theorem, the

$$\text{hypotenuse} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

Therefore,

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{4}{5} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} = \frac{3}{4}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{3}{5} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} = \frac{5}{3}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{4}{3} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} = \frac{5}{4}$$

If we examine Table 2, we observe that the six trigonometric functions are related in a number of ways. For example,

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \quad \text{and} \quad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}} \tag{1}$$

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \quad \text{and} \quad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}} \tag{2}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \quad \text{and} \quad \cot\theta = \frac{\text{adjacent}}{\text{opposite}} \tag{3}$$

Thus, we see the functions in (1) as reciprocals of one another. The functions in (2) are similarly reciprocals, as are the functions in (3).

Furthermore, since

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \quad \text{and} \quad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}$$

we have that

$$\frac{\sin\theta}{\cos\theta} = \frac{\dfrac{\text{opposite}}{\text{hypotenuse}}}{\dfrac{\text{adjacent}}{\text{hypotenuse}}} = \frac{\text{opposite}}{\text{adjacent}} = \tan\theta \tag{4}$$

and similarly,

$$\frac{\cos\theta}{\sin\theta} = \cot\theta \tag{5}$$

If an equation is true for *all* values in the domain of the variable, then the equation is called an **identity**. The functional relationships in (1), (2), (3), (4) and (5) are true for any θ, where $0° < \theta < 90°$. Therefore, we may refer to these equations as trigonometric identities. These identities are summarized in Table 3.

TABLE 3 **Trigonometric Identities for $0° < \theta < 90°$**

$\csc\theta = \dfrac{1}{\sin\theta}$	$\sin\theta = \dfrac{1}{\csc\theta}$
$\sec\theta = \dfrac{1}{\cos\theta}$	$\cos\theta = \dfrac{1}{\sec\theta}$
$\cot\theta = \dfrac{1}{\tan\theta}$	$\tan\theta = \dfrac{1}{\cot\theta}$
$\dfrac{\sin\theta}{\cos\theta} = \tan\theta$	$\dfrac{\cos\theta}{\sin\theta} = \cot\theta$

Verify the calculations in Example 1 using these identities.

EXAMPLE 2 RIGHT TRIANGLE TRIGONOMETRY

Find the values of the trigonometric functions of 45°.

SOLUTION

Since the sum of the two acute angles in a right triangle must be 90°, if one angle is 45°, then the other must be 45° as well. Two equal angles in a triangle imply that the triangle is isosceles. Since the size of the triangle does not matter when evaluating the six trigonometric functions, we choose the two equal legs of our isosceles right triangle to be of length 1, as shown in Figure 17. From the Pythagorean Theorem, we see that the

$$\text{hypotenuse} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

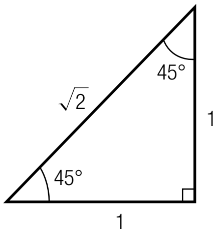


FIGURE 17 Diagram for Example 2

Therefore,

$$\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \qquad \cot 45° = 1$$

$$\cos 45° = \frac{\sqrt{2}}{2} \qquad \sec 45° = \sqrt{2}$$

$$\tan 45° = 1 \qquad \csc 45° = \sqrt{2}$$

EXAMPLE 3 RIGHT TRIANGLE TRIGONOMETRY

Find the values of the trigonometric functions of 30° and 60°.

SOLUTION

Consider equilateral triangle *ABD* with each side of length 2, as shown in Figure 18. Draw the angle bisector of angle *B* and let it intersect *AD* at point *C*.

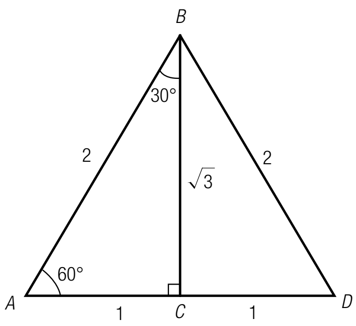


FIGURE 18 Diagram for Example 3

From plane geometry, we know that $\overline{AC} = \overline{CD} = 1$ and that *BC* is perpendicular to *AD*. Therefore, *ABC* is a right triangle, where angle *A* is 60° and angle *ABC* is 30°. From the Pythagorean Theorem, we calculate

$$\overline{BC} = \sqrt{2^2 - 1^2} = \sqrt{3}$$

Therefore,

$$\sin 30° = \frac{1}{2} \qquad \cot 30° = \sqrt{3}$$

$$\cos 30° = \frac{\sqrt{3}}{2} \qquad \sec 30° = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\tan 30° = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} \qquad \csc 30° = 2$$

and

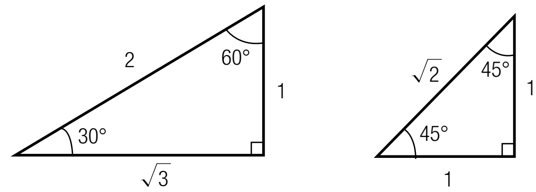
$$\sin 60° = \frac{\sqrt{3}}{2} \qquad \cot 60° = \frac{\sqrt{3}}{3}$$

$$\cos 60° = \frac{1}{2} \qquad \sec 60° = 2$$

$$\tan 60° = \sqrt{3} \qquad \csc 60° = \frac{2\sqrt{3}}{3}$$

Since we have already presented formulas for evaluating $\tan\theta$, $\cot\theta$, $\sec\theta$, and $\csc\theta$ in terms of $\sin\theta$ and $\cos\theta$, we summarize in Table 4 the values of sin θ and $\cos\theta$ for the "special angles" of Examples 2 and 3.

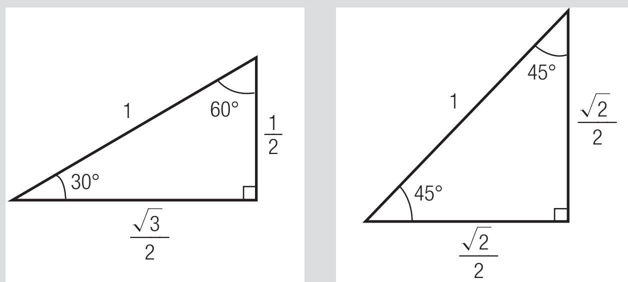
TABLE 4 sin θ and cos θ for 30°, 45°, and 60°



θ in Degrees	θ in Radians	sin θ	cos θ
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$

✔ Progress Check

Verify that each of the following triangles is similar to one of the triangles found in Table 4. Show that the corresponding sides are proportional. Furthermore, verify that the entries for sin θ and cos θ for the following triangles are identical to those found in Table 4.



If we examine the values of the trigonometric functions in Example 3, we observe that the same six numbers appear in the answers for 30° and for 60°, although in a different order. We will now establish the relationships causing these repetitions by considering triangle *ABC* in Figure 19.

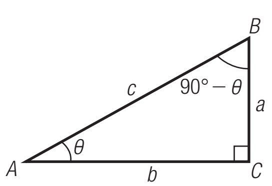


FIGURE 19 Right Triangle *ABC*

$$\sin\theta = \frac{a}{c} = \cos(90° - \theta) \tag{6}$$

$$\cos\theta = \frac{b}{c} = \sin(90° - \theta) \tag{7}$$

$$\tan\theta = \frac{a}{b} = \cot(90° - \theta) \tag{8}$$

$$\cot\theta = \frac{b}{a} = \tan(90° - \theta) \tag{9}$$

$$\sec\theta = \tfrac{c}{b} = \csc(90° - \theta) \qquad (10)$$

$$\csc\theta = \tfrac{c}{a} = \sec(90° - \theta) \qquad (11)$$

Angles θ and $90° - \theta$ are called **complementary** since their sum is 90°. In relationship (6), we see that the cosine of the angle complementary to θ is the sine of θ. In (7), the sine of the angle complementary to θ is the cosine of θ. Therefore, we say that sine and cosine are **cofunctions**. The origin of the prefix "co" used in cosine and cofunction is from the first two letters of the word "complement." In a similar manner, tangent and cotangent are cofunctions from (8) and (9), as are secant and cosecant from (10) and (11).

Verify the calculations in Examples 2 and 3 using the cofunction relationships.

Graphing Calculator Alert

When you turn on a scientific calculator, the display indicates one of the following modes:

DEG RAD GRAD

which are abbreviations for degree, radian, and grad, respectively. To ensure that our arguments are treated as real numbers, the display must read RAD. To do this, find the key marked $\boxed{\text{DRG}}$. Pressing this key causes the setting to rotate among the modes. (*Note:* Some calculators may have a different procedure for setting the angle mode. Check your owner's manual for details.) These measurements are related as follows.

$$0° = 0 \text{ radians} = 0 \text{ grads}$$

$$90° = \frac{\pi}{2} \text{ radians} = 100 \text{ grads}$$

$$180° = \pi \text{ radians} = 200 \text{ grads}$$

$$270° = \frac{3\pi}{2} \text{ radians} = 300 \text{ grads}$$

$$360° = 2\pi \text{ radians} = 400 \text{ grads}$$

Since grads are used infrequently, we will work solely with degrees and radians.

Graphing Calculator Alert

For graphing calculators and other calculators that have a direct entry format, each key sequence should be followed by the key $\boxed{\text{ENTER}}$, $\boxed{\text{EXEC}}$, or $\boxed{=}$, depending on your calculator, to display the answer. Consider the following examples.

Set your calculator to DEG mode.

(a) $\sin 30° = 0.5$

$\boxed{\sin}$ $\boxed{30}$

(b) $\cos 45° \approx 0.7071067812$

$\boxed{\cos}$ $\boxed{45}$

(Verify that this is the same answer as $\frac{\sqrt{2}}{2}$.)

Set your calculator to RAD mode. Use the $\boxed{\pi}$ key for these computations.

(c) $\sin \frac{\pi}{6} = 0.5$

$\boxed{\sin}$ $\boxed{(}$ $\boxed{\pi}$ $\boxed{\div}$ $\boxed{6}$ $\boxed{)}$

(d) $\cos 0.1 \approx 0.9950041653$

$\boxed{\cos}$ $\boxed{.}$ $\boxed{1}$

(e) $\tan \frac{\pi}{4} = 1$

$\boxed{\tan}$ $\boxed{(}$ $\boxed{\pi}$ $\boxed{\div}$ $\boxed{4}$ $\boxed{)}$

(f) $\sec \frac{\pi}{3} = 2$

$\boxed{1}$ $\boxed{\div}$ $\boxed{\cos}$ $\boxed{(}$ $\boxed{\pi}$ $\boxed{\div}$ $\boxed{3}$ $\boxed{)}$

or

$\boxed{(}$ $\boxed{\cos}$ $\boxed{(}$ $\boxed{\pi}$ $\boxed{\div}$ $\boxed{3}$ $\boxed{)}$ $\boxed{)}$ $\boxed{x^{-1}}$

Solving a Triangle

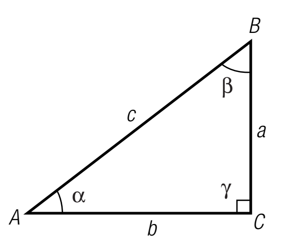


FIGURE 20 Notation for Parts of a Triangle

The expression "to solve a triangle" is used to indicate that we seek all parts of the triangle, that is, the length of each side and the measure of each angle. For any right triangle, given any two sides, or given one side and an acute angle, it is always possible to solve the triangle. We will standardize our notation as shown in Figure 20, namely, we identify the measure of the angles at A, B, and C by α, β, and γ, respectively. Furthermore, the notation for the lengths of the sides opposite A, B, and C are a, b, and c, respectively, as defined earlier in this section.

EXAMPLE 4 SOLVING A TRIANGLE

In Triangle ABC, $\beta = 27°$, $\gamma = 90°$, and $b = 8.6$, as shown in Figure 21. Find approximate values for the remaining parts of the triangle using a calculator.

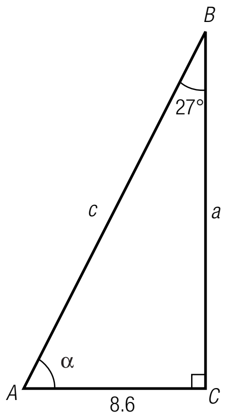


FIGURE 21 Diagram for Example 4

SOLUTION

Since $\alpha + 27° = 90°$, $\alpha = 63°$. To find c, we may consider

$$\sin 27° = \frac{8.6}{c}$$

in which case

$$c = \frac{8.6}{\sin 27°} \approx \frac{8.6}{0.4540} \approx 18.94$$

To find a, consider

$$\tan 27° = \frac{8.6}{a}$$

in which case

$$a = \frac{8.6}{\tan 27°} \approx \frac{8.6}{0.5095} \approx 16.88$$

(Verify the answers to a and c using trigonometric functions and 63°.)

✔ Progress Check

In triangle ABC, $\alpha = 64°$, $\gamma = 90°$, and $b = 24.7$. Solve the triangle.

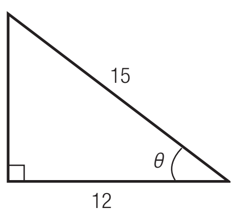
Answer

$\beta = 26°$, $a = 50.64$, $c = 56.34$

Exercise Set 7.2

In Exercises 1–8, find the values of the trigonometric functions of the angle θ.

1.



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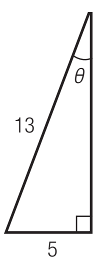
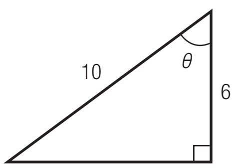


3.



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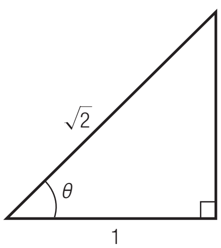

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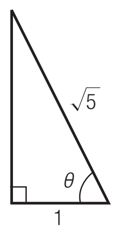

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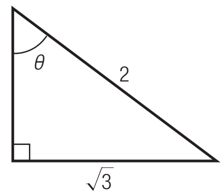

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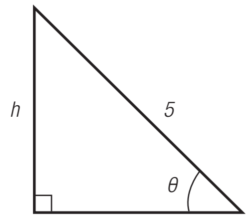

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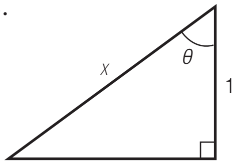
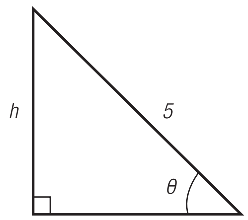
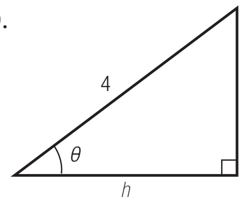


In Exercises 9–14, express the length b using a trigonometric function of the angle θ.

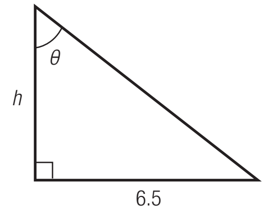
9.



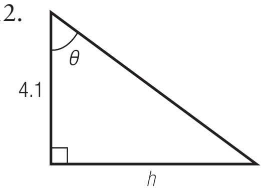
10.



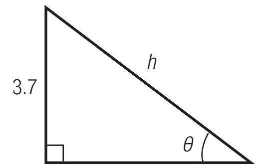
11.



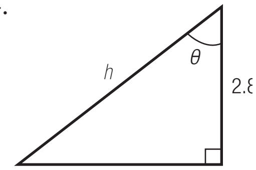
12.



13.



14.



In Exercises 15–20, we are given triangle ABC with $\gamma = 90°$. Find approximate values for the remaining parts of the triangle.

15. $b = 40$, $\beta = 40°$; find c.

16. $a = 22$, $\alpha = 36°$; find b.

17. $a = 75$, $\beta = 22°$; find b.

18. $b = 60$, $\alpha = 53°$; find c.

19. $a = 25$, $\beta = 42°30'$; find c.

20. $b = 50$, $\alpha = 36°20'$; find a.

7.3 The Trigonometric Functions

In this section, we extend the definitions of the six trigonometric functions to include angles that may take on any value. However, it is important that these extensions retain their previous meaning for angles between 0° and 90°. Furthermore, since the trigonometric functions can all be written in terms of sine and cosine, we shall specifically focus on these two functions.

Let us begin by considering the coordinate axes with point P having coordinates (x, y) in the first quadrant, as shown in Figure 22. We denote the distance from the origin O to point P by r, and we form right triangle OPQ, where the coordinates of Q are $(x, 0)$. This enables us to calculate any trigonometric functions of θ. Specifically, observe that

$$\cos\theta = \frac{x}{r} \qquad \text{and} \qquad \sin\theta = \frac{y}{r}$$

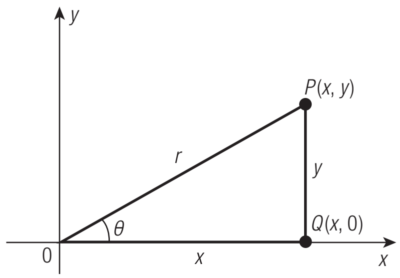


FIGURE 22 Right Triangle OPQ

Furthermore, the Pythagorean Theorem states that

$$x^2 + y^2 = r^2 \tag{1}$$

Dividing both sides by r^2, we obtain

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

In this case, Equation (1) is equivalent to

$$(\cos\theta)^2 + (\sin\theta)^2 = 1 \tag{2}$$

Now, suppose we fix the values of θ and change the value of r to r', so that r' is the distance from O to $P'(x', y')$, as shown in Figure 23. If Q' has coordinates $(x', 0)$, then triangle $OP'Q'$ is similar to triangle OPQ. For triangle $OP'Q'$,

$$\cos\theta = \frac{x'}{r'} \qquad \text{and} \qquad \sin\theta = \frac{y'}{r'}$$

However, as we already discussed in Section 7.2,

$$\frac{x'}{r'} = \frac{x}{r} \qquad \text{and} \qquad \frac{y'}{r'} = \frac{y}{r}$$

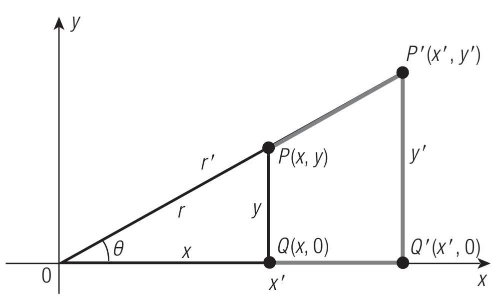


FIGURE 23 Right Triangles OPQ and $OP'Q'$

Therefore, since the "size of the triangle," or equivalently, the value for r, does not matter in determining $\sin\theta$ or $\cos\theta$, we choose $r = 1$, as shown in Figure 24. We observe that

$$\cos\theta = x \qquad \text{and} \qquad \sin\theta = y$$

Alternatively, we may say that the coordinates of point P are $(\cos\theta, \sin\theta)$. Furthermore, the Pythagorean Theorem states that

$$x^2 + y^2 = 1 \tag{3}$$

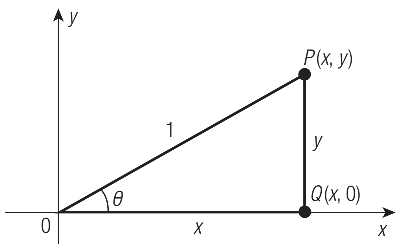


FIGURE 24 Right Triangle OPQ

In this case, Equation (3) is equivalent to

$$(\cos\theta)^2 + (\sin\theta)^2 = 1 \tag{4}$$

Note that Equation (4) is the same equation as (2), which we obtained from Figure 22 when $\overline{OP} = r$.

The graph of $x^2 + y^2 = 1$ is a circle centered at the origin with radius 1. We call this graph the **unit circle.** (Refer to Sections 3.1 and 5.2 for additional information concerning the equation of a circle and its symmetries.) Since the coordinates of the unit circle in the first quadrant correspond to $(\cos\theta, \sin\theta)$, it might seem reasonable to extend the definitions of $\cos\theta$ and $\sin\theta$ using $(\cos\theta, \sin\theta)$ for points on the unit circle in the remaining three quadrants as well. In fact, that is precisely what we will do. (Because the trigonometric functions are so closely related to the unit circle, they are sometimes referred to as **circular functions.**)

The circumference of the unit circle is $2\pi r = 2\pi(1) = 2\pi$. Choose a value t with $0 \le t < 2\pi$, and sweep out $\overset{\frown}{QP}$ of length t in the counterclockwise direction, as shown in Figure 25. If θ is measured in radians, then we may use the formula relating θ to the length of the arc s subtended by θ.

$$\theta = \frac{s}{r} = \frac{t}{1} = t$$

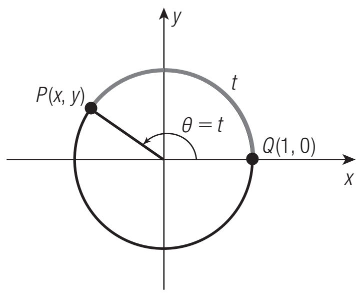


FIGURE 25 Choosing t on the Unit Circle, $0 \le t < 2\pi$

Therefore, $0 \le \theta < 2\pi$ and $P(x, y)$ is the unique point on the unit circle determined either by the length of arc t or by the angle $\theta = t$ radians. We let $P(t)$ denote the point on the unit circle determined by t, and we write

$$P(t) = P(x, y)$$

to indicate that the rectangular coordinates of the point on the unit circle are (x, y). We now define

$$\cos t = x \qquad \text{and} \qquad \sin t = y \qquad \text{for } 0 \le t < 2\pi$$

Equivalently,

$$P(t) = P(x, y) = P(\cos t, \sin t) \qquad \text{for } 0 \le t < 2\pi$$

EXAMPLE 1 FINDING COORDINATES OF POINTS ON THE UNIT CIRCLE

Find the coordinates of the point $P(t)$ on the unit circle for each of the following.

(a) $t = 0$ (b) $t = \dfrac{\pi}{2}$ (c) $t = \pi$ (d) $t = \dfrac{3\pi}{2}$

SOLUTION

(a) When $t = 0$, $P(0)$ and $Q(1, 0)$ represent the same point. Therefore, as we see in Figure 26(a),

$$P(0) = P(1, 0)$$

(b) Since

$$\frac{\frac{\pi}{2}}{2\pi} = \frac{1}{4}$$

$\frac{\pi}{2}$ represents one-quarter of the circumference of the circle. Therefore, as we see in Figure 26(b),

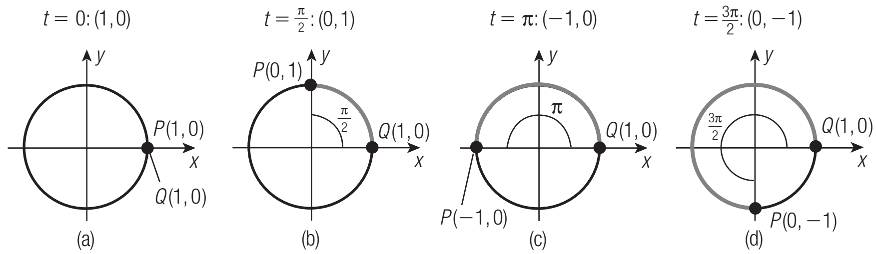


FIGURE 26 Coordinates of Points on the Unit Circle

$$P\left(\frac{\pi}{2}\right) = P(0, 1)$$

(c) Since

$$\frac{\pi}{2\pi} = \frac{1}{2}$$

π represents one-half of the circumference of the circle. As we see in Figure 26(c),

$$P(\pi) = P(-1, 0)$$

(d) Since

$$\frac{\frac{3\pi}{2}}{2\pi} = \frac{3}{4}$$

$\frac{3\pi}{2}$ represents three-quarters of the circumference of the circle. As we see in Figure 26(d),

$$P\left(\frac{3\pi}{2}\right) = P(0, -1)$$

EXAMPLE 2 FINDING COS *t* AND SIN *t*

Find $\cos t$ and $\sin t$ for the following values of t.

(a) $t = 0$ (b) $t = \frac{\pi}{2}$ (c) $t = \pi$ (d) $t = \frac{3\pi}{2}$

SOLUTION

From Example 1, we know that $P(0) = (1, 0)$, $P(\frac{\pi}{2}) = (0, 1)$, $P(\pi) = (-1, 0)$, and $P(\frac{3\pi}{2}) = (0, -1)$. Since we have $x = \cos t$ and $y = \sin t$ on the unit circle, it follows that

t	$\cos t$	$\sin t$
0	1	0
$\frac{\pi}{2}$	0	1
π	-1	0
$\frac{3\pi}{2}$	0	-1

Using the trigonometric identities from Section 7.2, Table 3, we now define the four remaining trigonometric functions of t in Table 5.

TABLE 5 tan t, cot t, sec t, csc t

$\tan t = \dfrac{\sin t}{\cos t}$,	$\cos t \neq 0$
$\cot t = \dfrac{\cos t}{\sin t}$,	$\sin t \neq 0$
$\sec t = \dfrac{1}{\cos t}$,	$\cos t \neq 0$
$\csc t = \dfrac{1}{\sin t}$,	$\sin t \neq 0$

EXAMPLE 3 FINDING TAN t, COT t, SEC t, AND CSC t

Find tan t, cot t, sec t, and csc t for the following values of t.

(a) $t = 0$ (b) $t = \dfrac{\pi}{2}$ (c) $t = \pi$ (d) $t = \dfrac{3\pi}{2}$

SOLUTION

Using the results of Example 2, we obtain

t	tan t	cot t	sec t	csc t
0	0	undefined	1	undefined
$\dfrac{\pi}{2}$	undefined	0	undefined	1
π	0	undefined	-1	undefined
$\dfrac{3\pi}{2}$	undefined	0	undefined	-1

(Verify that tan $\frac{\pi}{2}$ produces an error message on your calculator.)

Our goal is to define sin t and cos t for *all* values of t. To accomplish this, let us consider $t \geq 2\pi$. We sweep out the arc of length t in the counterclockwise direction, or equivalently, the angle $\theta = t$ radians, making as many revolutions as are needed. Since each revolution sweeps out an arc of length 2π and returns to the initial point $Q(1, 0)$, we can "ignore" all multiples of 2π. For example, if $t = \frac{5\pi}{2}$, we can write

$$t = \frac{5\pi}{2} = 2\pi + \frac{\pi}{2}$$

Sweeping out an arc of length 2π brings us back to $Q(1, 0)$. Then, we sweep out an arc of length $\frac{\pi}{2}$, arriving at the point $P(0, 1)$. To illustrate this, we use $\theta = \frac{5\pi}{2}$ in Figure 27. Therefore, $P(0, 1)$ corresponds to both $t = \frac{5\pi}{2}$ and $t = \frac{\pi}{2}$, or equivalently,

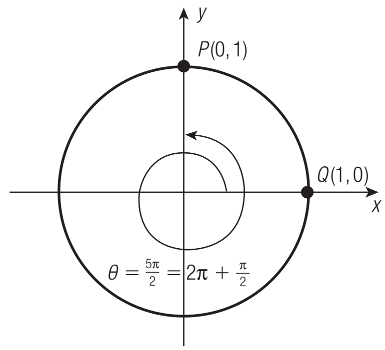


FIGURE 27 The Unit Circle

$$P\left(\frac{5\pi}{2}\right) = P\left(\frac{\pi}{2}\right) = P(0, 1)$$

Since $P(\frac{\pi}{2}) = (\cos \frac{\pi}{2}, \sin \frac{\pi}{2})$, we define $P(\frac{5\pi}{2}) = (\cos \frac{5\pi}{2}, \sin \frac{5\pi}{2})$ so that

$$\cos \frac{5\pi}{2} = \cos \frac{\pi}{2} \qquad \text{and} \qquad \sin \frac{5\pi}{2} = \sin \frac{\pi}{2}$$

Similarly, if $t = \frac{19\pi}{4}$, we can write

$$\frac{19\pi}{4} = 2\pi + 2\pi + \frac{3\pi}{4}$$

Therefore, as shown in Figure 28,

$$P\left(\frac{19\pi}{4}\right) = P\left(\frac{3\pi}{4}\right)$$

and hence

$$\left(\cos \frac{19\pi}{4}, \sin \frac{19\pi}{4}\right) = \left(\cos \frac{3\pi}{4}, \sin \frac{3\pi}{4}\right)$$

or

$$\cos \frac{19\pi}{4} = \cos \frac{3\pi}{4} \qquad \text{and} \qquad \sin \frac{19\pi}{4} = \sin \frac{3\pi}{4}$$

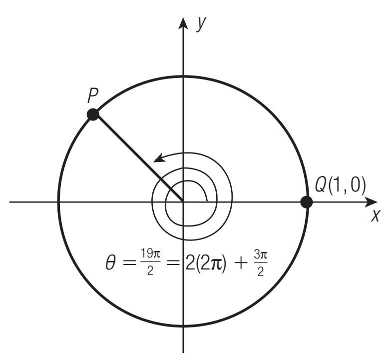


FIGURE 28 The Unit Circle

Finally, if we let $t < 0$, we sweep out the arc of length $|t|$ in the *clockwise* direction, or equivalently, the angle $\theta = t$. For example, if $t = -\frac{3\pi}{2}$, sweeping out an arc of length $\frac{3\pi}{2}$ in the clockwise direction takes us to $P(0, 1)$. We illustrate this in Figure 29 using $\theta = -\frac{3\pi}{2}$ and the arc of length $\frac{3\pi}{2}$. Therefore, $P(0, 1)$ corresponds to $t = -\frac{3\pi}{2}$ as well as to $t = \frac{\pi}{2}$, or equivalently,

$$P\left(-\frac{3\pi}{2}\right) = P\left(\frac{\pi}{2}\right) = P(0, 1)$$

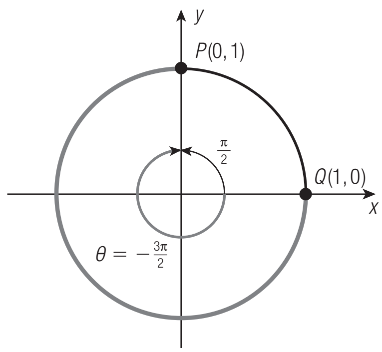


FIGURE 29 The Unit Circle

Since $P(\frac{\pi}{2}) = (\cos \frac{\pi}{2}, \sin \frac{\pi}{2})$, we define $P(-\frac{3\pi}{2}) = (\cos(-\frac{3\pi}{2}), \sin(-\frac{3\pi}{2}))$ so that

$$\cos\left(-\frac{3\pi}{2}\right) = \cos\frac{\pi}{2} \qquad \text{and} \qquad \sin\left(-\frac{3\pi}{2}\right) = \sin\frac{\pi}{2}$$

We have established the following:

For every real number t, there is a unique point $P(t)$ on the unit circle that can be found by sweeping out an arc of length $|t|$ in

1. the counterclockwise direction when $t \geq 0$
2. the clockwise direction when $t < 0$

Equivalently, the point $P(t)$ is uniquely determined by the angle $\theta = t$ radians. Furthermore, if

$$P(t) = P(x, y)$$

then we define

$$\cos t = x \qquad \text{and} \qquad \sin t = y \qquad \text{for any number } t$$

Equivalently,

$$P(t) = P(x, y) = P(\cos t, \sin t) \qquad \text{for any number } t$$

Note that $P(t)$, $\cos t$ and $\sin t$ are not one-to-one functions since

$$P\left(\frac{\pi}{2}\right) = P\left(\frac{5\pi}{2}\right) = P\left(-\frac{3\pi}{2}\right)$$

$$\cos\frac{\pi}{2} = \cos\frac{5\pi}{2} = \cos\left(-\frac{3\pi}{2}\right)$$

$$\sin\frac{\pi}{2} = \sin\frac{5\pi}{2} = \sin\left(-\frac{3\pi}{2}\right)$$

Since the circumference of the unit circle is 2π, or equivalently, since there are 2π radians in a circle, every complete revolution *in either direction* returns to the point $Q(1, 0)$. Therefore,

For every integer n,

$$P(t) = P(t + 2\pi n)$$

$$\sin t = \sin(t + 2\pi n)$$

$$\cos t = \cos(t + 2\pi n)$$

EXAMPLE 4 FINDING POINTS ON THE UNIT CIRCLE

Find the point on the unit circle corresponding to each of the following.

(a) $t = 6\pi$ (b) $t = 5\pi$ (c) $t = -\frac{11\pi}{2}$

SOLUTION

(a) Since $t = 6\pi = 3(2\pi)$, we must complete three revolutions, returning to the point on the unit circle $(1, 0)$. Thus, $P(6\pi) = P(1, 0)$.

(b) If we write

$$t = 5\pi = 2(2\pi) + \pi$$

we then need only sweep out an arc of length π to arrive at the point $(-1, 0)$. Thus, $P(5\pi) = P(-1, 0)$.

(c) We write this in the form

$$t = -\frac{11\pi}{2} = -2(2\pi) - \frac{3\pi}{2}$$

Discarding the multiples of 2π, we sweep out an arc of length $\frac{3\pi}{2}$ in a clockwise direction. This takes us to the point $(0, 1)$. Thus, $P(-\frac{11\pi}{2}) = P(0, 1)$.

✔ Progress Check

Find the point on the unit circle corresponding to

(a) $t = -\dfrac{15\pi}{2}$ (b) $t = -3\pi$ (c) $t = -\dfrac{\pi}{2}$

Answer

(a) $P(0,\ 1)$ (b) $P(-1, 0)$ (c) $P(0, -1)$

We now extend the definitions of tan t, cot t, sec t, and csc t to all values of t that are consistent with the previous definitions. In Table 6, we present these definitions, including the corresponding domains for each of the six trigonometric functions. Note that the domains corresponding to tan t, cot t, sec t, and csc t can be found as a consequence of the results of Example 3 and the fact that $\sin t = \sin\,(t + 2\pi n)$ and $\cos t = \cos\,(t + 2\pi n)$ for any integer n.

TABLE 6 Domains of the Trigonometric Functions

Function	Domain
$\sin t$	All values of t
$\cos t$	All values of t
$\tan t = \dfrac{\sin t}{\cos t}$	All values of $t \neq \frac{\pi}{2} + n\pi$ for any integer n
$\cot t = \dfrac{\cos t}{\sin t}$	All values of $t \neq n\pi$ for any integer n
$\sec t = \dfrac{1}{\cos t}$	All values of $t \neq \frac{\pi}{2} + n\pi$ for any integer n
$\csc t = \dfrac{1}{\sin t}$	All values of $t \neq n\pi$ for any integer n

Graphing Calculator Power User's Corner 7.3

A Common Source of Confusion

When students first begin working with trigonometric functions, one common mistake is to assume that to find the value of the cosecant of an angle one should type [2$^{\text{nd}}$][sin]. THIS IS NOT THE CASE. Typing [2$^{\text{nd}}$][sin] finds the arcsine—the inverse of the sine function. Similarly, [2$^{\text{nd}}$][cos] and [2$^{\text{nd}}$][tan] find the arccosine and arctangent.

In order to find the value of the cosecant of an angle such as 2.09 radians, begin by considering the sine of that angle. Using the calculator, you would type [sin]2.09[enter] and get .868.

Bearing in mind that the cosecant is the reciprocal of the sine, you find the cosecant of the same angle by typing 1/[sin]2.09[enter] to get the correct answer of 1.152.

When working with secant, type 1/[cos] and then the angle measure.

When working with cotangent, type 1/[tan] and then the angle measure.

Practice Problems

Use your calculator to find the following. (All angles are in radians unless otherwise specified.)

1. sin .785 csc .785
2. cos 2.749 sec 2.749
3. tan 5.760 cot 5.760

Exercise Set 7.3

In Exercises 1–8, determine the quadrant in which t lies.

1. $t = \frac{4\pi}{3}$
2. $t = -\frac{\pi}{6}$
3. $t = \frac{5\pi}{6}$
4. $t = \frac{5\pi}{4}$
5. $t = -\frac{19\pi}{18}$
6. $t = \frac{3\pi}{4}$
7. $t = -\frac{7\pi}{6}$
8. $t = \frac{10\pi}{6}$

In Exercises 9–20, for each given real number s, find a real number t in the interval $[0, 2\pi)$ so that the point on the unit circle $P(t) = P(s)$.

9. 4π
10. $\frac{13\pi}{2}$
11. $\frac{15\pi}{7}$
12. $-\frac{25\pi}{4}$
13. $-\frac{21\pi}{2}$
14. $-\frac{11\pi}{2}$
15. $\frac{41\pi}{6}$
16. $\frac{11\pi}{2}$
17. -9π
18. 7π
19. $\frac{27\pi}{5}$
20. $-\frac{22\pi}{3}$

In Exercises 21–24, plot the approximate positions of the points on the unit circle.

21. $P(7\pi), P\left(\frac{4\pi}{3}\right), P\left(\frac{5\pi}{2}\right), P\left(-\frac{7\pi}{4}\right), P\left(-\frac{13\pi}{6}\right)$
22. $P\left(\frac{11\pi}{2}\right), P\left(-\frac{3\pi}{4}\right), P\left(\frac{11\pi}{6}\right), P\left(-\frac{10\pi}{3}\right), P\left(\frac{33\pi}{4}\right)$
23. $P(-10), P(8), P(3.3), P(-4), P(1.7)$
24. $P(14), P(-0.5), P(-8), P(6), P(3)$

In Exercises 25–36, find the point on the unit circle corresponding to each of the following.

25. $t = 3\pi$
26. $t = 12\pi$
27. $t = \frac{7\pi}{2}$
28. $t = \frac{15\pi}{2}$
29. $t = \frac{23\pi}{2}$
30. $t = -3\pi$
31. $t = -18\pi$
32. $t = -\frac{7\pi}{2}$
33. $t = -\frac{15\pi}{2}$
34. $t = -\frac{23\pi}{2}$
35. $t = \frac{17\pi}{2}$
36. $t = -\frac{17\pi}{2}$

In Exercises 37–44, $P = P(x, y) = P(t)$ is a point on the unit circle. Find $\sin t$ and $\cos t$.

37. $P\left(-\frac{3}{5}, \frac{4}{5}\right)$
38. $P\left(-\frac{4}{5}, \frac{3}{5}\right)$
39. $P\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
40. $P\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

41. $P\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

42. $P\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

43. $P\left(\frac{\sqrt{15}}{4}, -\frac{1}{4}\right)$

44. $P\left(\frac{1}{4}, \frac{\sqrt{15}}{4}\right)$

In Exercises 45–52, find $\tan t$, $\cot t$, $\sec t$, and $\csc t$ for the points on the unit circle given in Exercises 37–44, respectively.

7.4 Special Values and Properties of Trigonometric Functions

In the previous section, we showed that for every real number t, there is a unique point

$$P(t) = (\cos t, \sin t)$$

on the unit circle. We have found the point $P(t)$ for $t = 0, \frac{\pi}{2}, \pi$, and $\frac{3\pi}{2}$. However, it is not easy to determine the point $P(t)$ for every given value of t. In this section, we shall find $P(t)$ by using right triangle trigonometry.

Since the unit circle is symmetric with respect to the x-axis, the y-axis, and the origin, these symmetries will be quite useful in evaluating the extensions of the trigonometric functions discussed previously. For example, if $\theta = \frac{\pi}{3}$ radians, the coordinates on the unit circle are $(\frac{1}{2}, \frac{\sqrt{3}}{2})$. (See Table 4.) Using the symmetries of the circle, we see that $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$, $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$, and $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$ are also points on the unit circle, as shown in Figure 30. However, what values of θ correspond to these points? Consider the unit circle in Figure 31. Since the angle swept out by the arc counterclockwise from Q to $(\frac{1}{2}, \frac{\sqrt{3}}{2})$ is $\frac{\pi}{3}$, using plane geometry, we calculate that the angle swept out by the arc counterclockwise from Q to $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ is $\frac{2\pi}{3}$. Then $\cos \frac{2\pi}{3} = -\frac{1}{2}$ and $\sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}$. In a similar fashion, we calculate $\cos \frac{4\pi}{3} = -\frac{1}{2}$, $\sin \frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$, $\cos \frac{5\pi}{3} = \frac{1}{2}$, and $\sin \frac{5\pi}{3} = -\frac{\sqrt{3}}{2}$. (Verify this.)

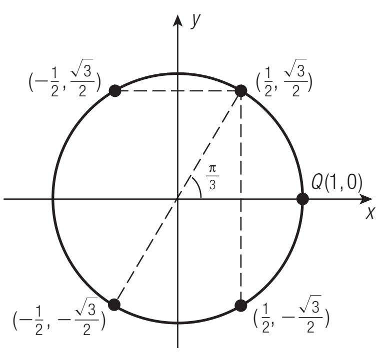


FIGURE 30 The Unit Circle

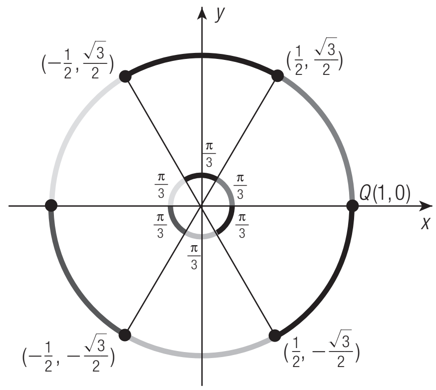


FIGURE 31 The Unit Circle

Recall the definition of a reference angle from Section 7.1 with associated Figure 10, which we present here as Figure 32. The reference angle θ' is acute.

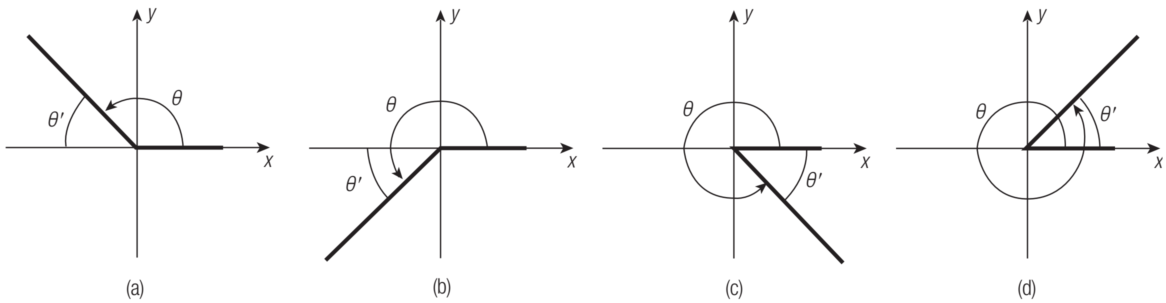


FIGURE 32 The Reference Angle

Therefore, we form a **reference right triangle**, having one leg on the x-axis, hypotenuse of length r, and an acute angle θ', as shown in Figure 33. The reference right triangle is called a **unit reference right triangle** if $r = 1$.

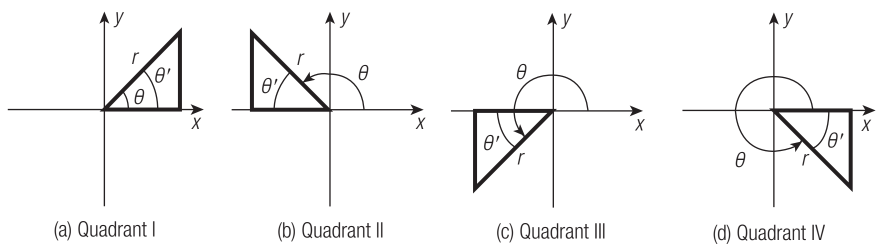


FIGURE 33 The Reference Right Triangle

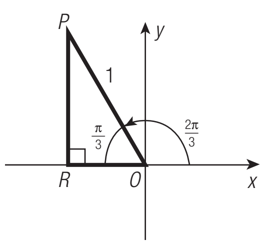


FIGURE 34 The Unit Reference Right Triangle for $\theta = {}^{2\pi}/_{3}$

Using the unit reference right triangle, we shall calculate $\cos \frac{2\pi}{3}$ and $\sin \frac{2\pi}{3}$. In Figure 34, we have the corresponding right triangle. From Table 4 in Section 7.2, we observe that in a 30°, 60°, 90° right triangle, or equivalently, a $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ radian right triangle, the side opposite the 30°, or $\frac{\pi}{6}$ radian, angle has length $\frac{1}{2}$ the length of the hypotenuse. Furthermore, the side opposite the 60°, or $\frac{\pi}{3}$ radian, angle has length $\frac{\sqrt{3}}{2}$ times the length of the hypotenuse. From this we determine that

$$\overline{OR} = \frac{1}{2} \qquad \text{and} \qquad \overline{RP} = \frac{\sqrt{3}}{2}$$

Now we are able to find the coordinates of P, namely, $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ from which we see that

$$\cos\frac{2\pi}{3} = -\frac{1}{2} \qquad \text{and} \qquad \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$$

The reference angle may also be defined in the case where $\theta < 0$. Consider the angles in Figure 35. Once again, the reference angle θ' corresponding to the angle θ is the acute angle formed by the terminal side of θ and the x-axis. If θ lies in the fourth quadrant, as in Figure 35(a), $\theta' = |\theta|$. The other cases are illustrated in Figures 35(b), 35(c). and 35(d). We form a reference right triangle using the same method described above.

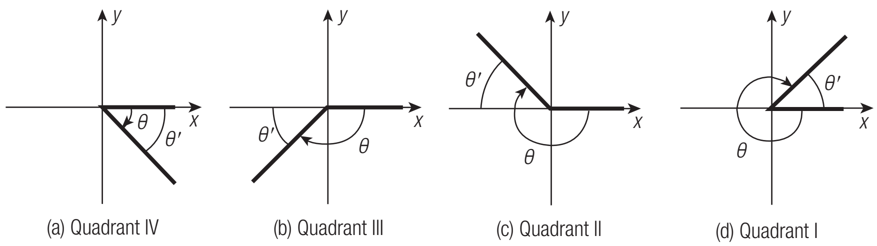


FIGURE 35 The Reference Angle

EXAMPLE 1 USING AN ISOSCELES UNIT REFERENCE RIGHT TRIANGLE

Find $\cos(-\frac{3\pi}{4})$ and $\sin(-\frac{3\pi}{4})$.

SOLUTION

Consider the unit reference right triangle in Figure 36. From Table 4, we observe that in an isosceles right triangle, each leg has length $\frac{\sqrt{2}}{2}$ times the length of the hypotenuse. Therefore,

$$\overline{OR} = \frac{\sqrt{2}}{2} \qquad \text{and} \qquad \overline{RP} = \frac{\sqrt{2}}{2}$$

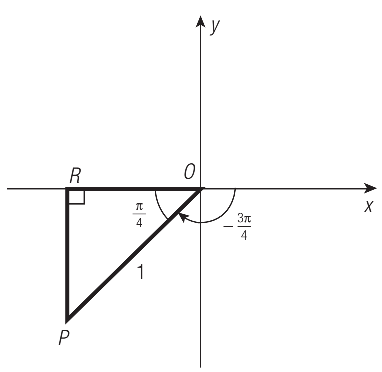


FIGURE 36 The Unit Reference Right Triangle for $\theta = -\dfrac{3\pi}{4}$

We are now able to find the coordinates of P, namely, $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$ from which we obtain

$$\cos\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2} \qquad \text{and} \qquad \sin\left(-\frac{3\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$

EXAMPLE 2 USING A UNIT REFERENCE RIGHT TRIANGLE

Find $\tan \frac{5\pi}{6}$.

SOLUTION

Using the properties of a 30°, 60°, 90° right triangle as shown in Figure 37, we find that $\overline{OR} = \frac{\sqrt{3}}{2}$ and $\overline{RP} = \frac{1}{2}$. Therefore, the coordinates of P are $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, hence, $\cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$ and $\sin \frac{5\pi}{6} = \frac{1}{2}$. Since $\tan t = \frac{\sin t}{\cos t}$, we have

$$\tan \frac{5\pi}{6} = \frac{\sin \frac{5\pi}{6}}{\cos \frac{5\pi}{6}} = \frac{\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

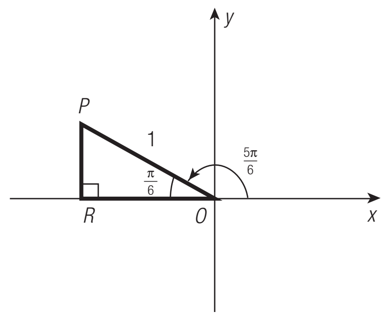


FIGURE 37 The Unit Reference Right Triangle for $\theta = \frac{5\pi}{6}$

✔ Progress Check

Find $\csc\left(-\frac{4\pi}{3}\right)$.

Answers

$\frac{2\sqrt{3}}{3}$

EXAMPLE 3 USING A UNIT REFERENCE RIGHT TRIANGLE

Find $\cos \theta$ and $\sin \theta$ if the point $(-4, 3)$ lies on the terminal side of θ.

SOLUTION

The reference angle of θ is θ'. Using the Pythagorean Theorem, we find that the length of the hypotenuse is

$$\sqrt{(-4^2) + (3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5$$

As shown in Figure 38, the line from the origin to $P'(-4, 3)$ intersects the unit circle at $P(-\frac{4}{5}, \frac{3}{5})$ since

$$\overline{OP} = \frac{1}{5}\overline{OP'} = \frac{1}{5}(5) = 1$$

Therefore,

$$\cos\theta = -\frac{4}{5} \qquad \text{and} \qquad \sin\theta = \frac{3}{5}$$

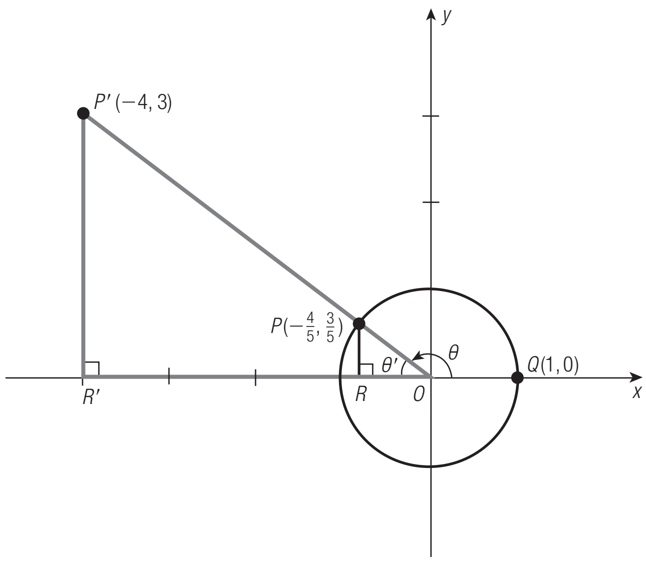


FIGURE 38 The Unit Reference Right Triangle for θ

In Example 3, we used a unit reference right triangle, namely, triangle OPR in Figure 38, to find $\cos\theta$ and $\sin\theta$. Specifically, using the coordinates of P, we had

$$P(\cos\theta, \sin\theta) = P\left(-\frac{4}{5}, \frac{3}{5}\right) \tag{1}$$

Hence

$$\cos\theta = -\frac{4}{5} \qquad \text{and} \qquad \sin\theta = \frac{3}{5}$$

Is it possible to find $\cos\theta$ and $\sin\theta$ from the coordinates of P'? Since right triangle OPR and right triangle $OP'R'$ both have reference angle θ', they are similar. Thus, their corresponding sides are proportional. Since $\overline{OP} = 1$ and $\overline{OP'} = 5$,

$$\overline{OP'} = 5\ \overline{OP}$$

in which case

$$\overline{OR'} = 5\ \overline{OR} \qquad \text{and} \qquad \overline{P'R'} = 5\ \overline{PR}$$

Thus, the coordinates of P' are 5 times the coordinates of P, respectively. From Equation (1),

$$P'(5\cos\theta, 5\sin\theta) = P'\left(5\left(-\frac{4}{5}\right), 5\left(\frac{3}{5}\right)\right)$$

$$P'(5\cos\theta, 5\sin\theta) = P'(-4, 3) \qquad (2)$$

Hence

$$5\cos\theta = -4 \qquad \text{and} \qquad 5\sin\theta = 3$$

Therefore, using the coordinates of P' from Equation (2), we are able to obtain

$$\cos\theta = -\frac{4}{5} \qquad \text{and} \qquad \sin\theta = \frac{3}{5}$$

In general, one may use a reference right triangle to find $\cos\theta$ and $\sin\theta$, not only a unit reference right triangle. Thus, as shown in Figure 39, if P has coordinates (x, y) and $r = \sqrt{x^2 + y^2} \neq 0$, then

$$P(r\cos\theta, r\sin\theta) = P(x, y) \qquad (3)$$

Hence

$$r\cos\theta = x \qquad \text{and} \qquad r\sin\theta = y$$

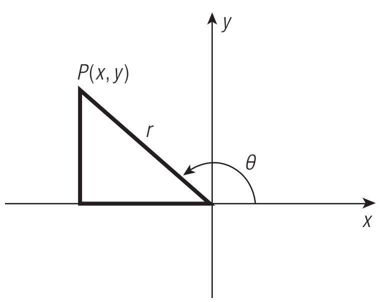


FIGURE 39 The Reference Right Triangle for θ

Therefore, using the coordinates of P from Equation (3), we are able to obtain

$$\cos\theta = \frac{x}{r} \qquad \text{and} \qquad \sin\theta = \frac{y}{r}$$

To summarize:

If P has coordinates (x, y) with $r = \sqrt{x^2 + y^2} \neq 0$, then

$$P(x, y) = P(r\cos\theta, r\sin\theta)$$

and

$$\cos\theta = \frac{x}{r}, \qquad \sin\theta = \frac{y}{r}$$

Observe that these are the same equations for $\cos\theta$ and $\sin\theta$ presented at the beginning of Section 7.3.

In Table 7, we summarize some special values of $\cos\theta$ and $\sin\theta$. We illustrate some of these values in Figure 40.

TABLE 7 Special Values of $\cos\theta$ and $\sin\theta$

θ	$\cos\theta$	$\sin\theta$	θ	$\cos\theta$	$\sin\theta$
0	1	0	π	-1	0
$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{7\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{4\pi}{3}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	0	1	$\frac{3\pi}{2}$	0	-1
$\frac{2\pi}{3}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{5\pi}{3}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$\frac{3\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{7\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$\frac{5\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{11\pi}{6}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
			2π	1	0

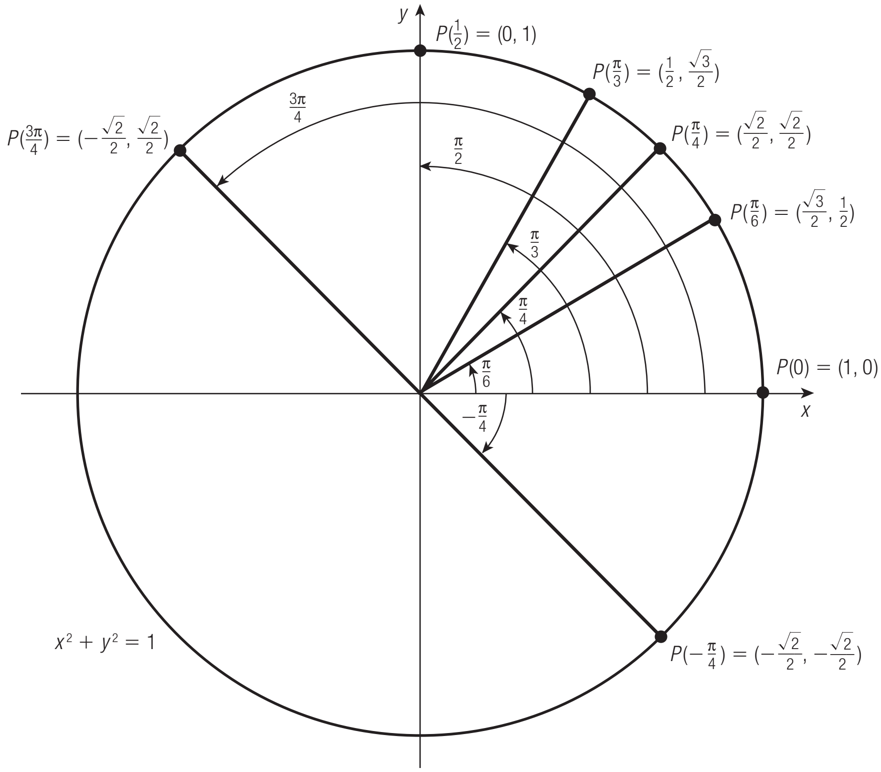


FIGURE 40 Special Values of cos θ and sin θ and the Unit Circle

Properties of the Trigonometric Functions

Consider Figure 41. Let point P be any point on the unit circle. If $P(x, y)$ is in the first or fourth quadrant, then $x > 0$, whereas if $P(x, y)$ is in the second or third quadrant, then $x < 0$. Since $\cos t = x$, we conclude that $\cos t > 0$ if t is in quadrant I or IV and $\cos t < 0$ if t is in quadrant II or III. Similarly, $y > 0$ if $P(x, y)$ is in quadrant I or II, whereas $y < 0$ if $P(x, y)$ is in quadrant III or IV. Since $\sin t = y$, we may write similar inequalities for $\sin t$. We present these results in Figure 42.

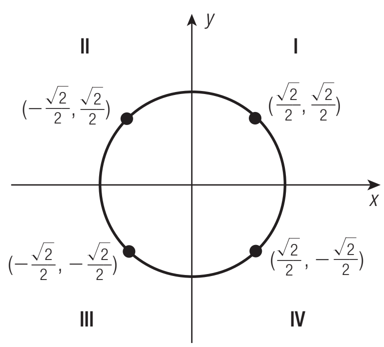


FIGURE 41 Points on the Unit Circle

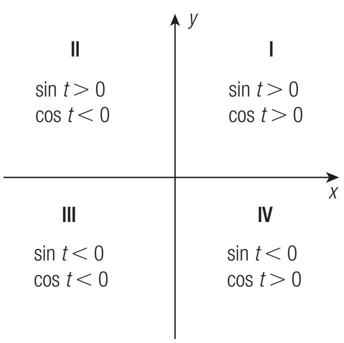


FIGURE 42 Signs of sin t and cos t

Since $\tan t = \frac{\sin t}{\cos t}$ if $\sin t$ and $\cos t$ have the same *sign*, $\tan t > 0$. If $\sin t$ and $\cos t$ have opposite signs, $\tan t < 0$. Using this with the other definitions presented in Table 6, Section 7.3, we summarize the signs for all the trigonometric functions in Figure 43. We have presented two columns in each quadrant of Figure 43. The order of the inequalities appearing in the first column of a given quadrant is the same as the order of the corresponding reciprocals appearing in the second column of the same quadrant. Therefore, we focus on the column listing $\sin t$, $\cos t$, and $\tan t$. Additionally, we present Figure 44, which identifies those quadrants where $\sin t$, $\cos t$, and $\tan t$ are positive. Hence, by inference, any of these three functions omitted in a particular quadrant is negative in that quadrant.

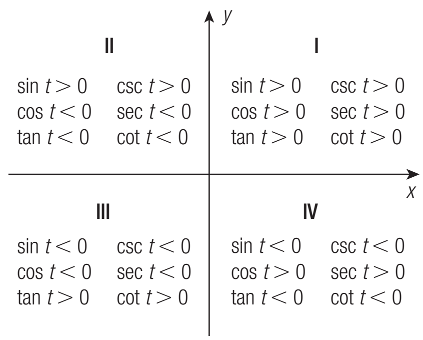


FIGURE 43 Signs of the Trigonometric Functions

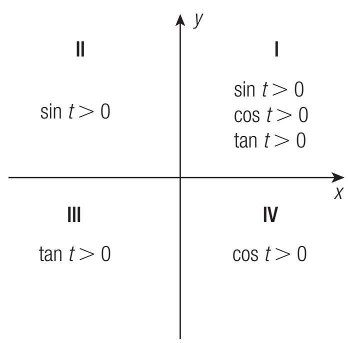


FIGURE 44 Signs of $\sin t$, $\cos t$, and $\tan t$

EXAMPLE 4 FINDING THE QUADRANT OF $P(t)$

Determine the quadrant in which $P(t)$ lies in each of the following.

(a) $\sin t > 0$ and $\tan t < 0$

(b) $\sin t < 0$ and $\cos t > 0$

(c) $\sec t > 0$ and $\csc t < 0$

SOLUTION

(a) Since $\sin t > 0$ in quadrants I and II and $\tan t < 0$ in quadrants II and IV, both conditions hold only in quadrant II.

(b) Since $\sin t < 0$ in quadrants III and IV and $\cos t > 0$ in quadrants I and IV, both conditions apply only in quadrant IV.

(c) Since $\sec t > 0$ in quadrants I and IV and $\csc t < 0$ in quadrants III and IV, both conditions apply only in quadrant IV.

✔ Progress Check

Determine the quadrant in which $P(t)$ lies in each of the following:

(a) $\cos t < 0$ and $\tan t > 0$
(b) $\cos t < 0$ and $\sin t > 0$
(c) $\tan t < 0$ and $\csc t > 0$

Answers

(a) quadrant III (b) quadrant II (c) quadrant II

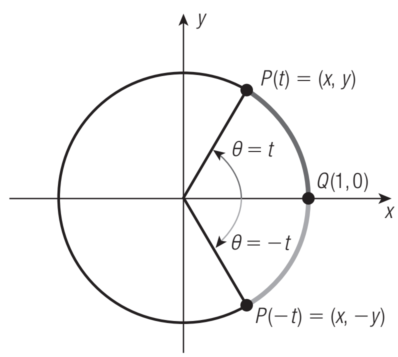


FIGURE 45 $P(t)$ and $P(-t)$ on a Unit Circle

We can use the symmetries of the unit circle to find $\sin(-t)$ and $\cos(-t)$. In Figure 45, we see that $P(t)$ and $P(-t)$ correspond to points having the same x-coordinates, whereas the y-coordinates are opposite in sign. Then

$$\sin t = y \qquad \text{and} \qquad \sin(-t) = -y$$

so

$$\sin(-t) = -\sin t$$

Similarly,

$$\cos t = x \qquad \text{and} \qquad \cos(-t) = x$$

so

$$\cos(-t) = \cos t$$

Using the definitions from Table 6, we obtain

$$\sin(-t) = -\sin t$$
$$\cos(-t) = \cos t$$
$$\tan(-t) = -\tan t$$
$$\cot(-t) = -\cot t$$
$$\sec(-t) = \sec t$$
$$\csc(-t) = -\sec t$$

A function f is said to be an **even function** if $f(-x) = f(x)$. We see that cosine and secant are even functions. A function f is said to be an **odd function** if $f(-x) = -f(x)$. The remaining four trigonometric functions are odd functions. In the case of polynomial functions, observe that $f(x) = x^2$ is an even function, but $f(x) = x^3$ is an odd function. From Section 3.3, we know that the graph of an even function is symmetric about the y-axis, and the graph of an odd function is symmetric with respect to the origin.

EXAMPLE 5 EVEN AND ODD FUNCTIONS

Find $\sin\left(-\frac{\pi}{4}\right)$ and $\cos\left(-\frac{\pi}{3}\right)$.

SOLUTION

$$\sin\left(-\frac{\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2} \qquad \cos\left(-\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$$

✔ Progress Check

Verify the results of Example 1 and the first Progress Check in this section using the formulas for $\sin(-t)$ and $\cos(-t)$.

Identities

Trigonometry often involves the use of identities, which are equations that are true for *all* values which may be assumed by the variable. Identities are useful in simplifying equations and in providing alternative forms for computations. Although we will devote an entire section to this topic in the next chapter, there are some basic identities that we present at this time. First we review the identities from Table 6, Section 7.3.

$$\tan t = \frac{\sin t}{\cos t} \qquad \sec t = \frac{1}{\cos t}$$

$$\cot t = \frac{\cos t}{\sin t} \qquad \csc t = \frac{1}{\sin t}$$

Next, observe that the coordinates (x, y) of every point on the unit circle satisfy the equation

$$x^2 + y^2 = 1$$

Since $x = \cos t$ and $y = \sin t$, we obtain

$$(\cos t)^2 + (\sin t)^2 = 1$$

(See Equation (4) of Section 7.3.)

Since powers of trigonometric functions, such as $(\sin t)^n$ and $(\cos t)^n$, occur quite frequently, a special notation is used. For example, we write

$$(\sin t)^n = \sin^n t \qquad \text{and} \qquad (\cos t)^n = \cos^n t \qquad \text{provided } n \neq -1$$

Using the notation and rearranging terms, the identity becomes

$$\sin^2 t + \cos^2 t = 1$$

We may also use this identity in the alternative forms

$$\sin^2 t = 1 - \cos^2 t$$
$$\cos^2 t = 1 - \sin^2 t$$

WARNING

Do not confuse

$$\sin t^2 \qquad \text{and} \qquad \sin^2 t$$

We have defined $\sin^2 t$ by

$$\sin^2 t = (\sin t)^2$$

which indicates that we find $\sin t$ and then square the result. But $\sin t^2$ indicates that we are to square t and *then* find the sine of the argument t^2.

EXAMPLE 6 APPLYING THE TRIGONOMETRIC IDENTITIES

If $\cos t = \frac{3}{5}$ and t is in quadrant IV, find $\sin t$ and $\tan t$.

SOLUTION

Using the identity $\sin^2 t + \cos^2 t = 1$, we have

$$\sin^2 t + \left(\frac{3}{5}\right)^2 = 1$$

$$\sin^2 t = 1 - \frac{9}{25} = \frac{16}{25}$$

$$\sin t = \pm\frac{4}{5}$$

Since t is in quadrant IV, $\sin t$ must be negative so that $\sin t = -\frac{4}{5}$. Then

$$\tan t = \frac{\sin t}{\cos t} = \frac{-\frac{4}{5}}{\frac{3}{5}} = -\frac{4}{3}$$

✔ Progress Check

If $\sin t = \frac{12}{13}$ and t is in quadrant II, find the following:

(a) $\cos t$ (b) $\tan t$

Answers

(a) $-\dfrac{5}{13}$ (b) $-\dfrac{12}{5}$

EXAMPLE 7 PROVING AN IDENTITY

Show that $1 + \tan^2 x = \sec^2 x$.

SOLUTION

We will use the trigonometric identities to transform the left-hand side of the equation into the right-hand side. Since $\tan x = \frac{\sin x}{\cos x}$, we have

$$1 + \tan^2 x = 1 + \frac{\sin^2 x}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

Since $\cos^2 x + \sin^2 x = 1$,

$$1 + \tan^2 x = \frac{1}{\cos^2 x} = \sec^2 x$$

Graphing Calculator Power User's Corner 7.4

Finding Non-identities Using the Calculator

Although you cannot use the calculator to prove a trigonometric identity, you can graph the two sides and see immediately if it is NOT an identity. For example, when confronted with $1 + \cos^2 x = \sin^2 x$, your calculator can show you quickly that this is an untrue statement using the following procedure.

Instruction	Screen View
Go to [Y=] and enter 1+(cos(x))[x^2]	Y_1=1+(cos(x))^2
Go to [Y=] and enter (sin(x))[x^2]	Y_2=(sin(x))^2
Now press [ZOOM] 7 to set the window automatically to trig mode, which goes from -2π to 2π on the x axis and from -4 to 4 on the y axis. This window is ideal for most trig graphs.	

You can see that the two sides of the equation are not equal since they produce different graphs.

Use the procedure above to determine if each of the following could be a true identity. Remember that even if the two graphs appear identical that is not sufficient evidence to say that is an identity. There may be non-examples that don't appear on the viewing screen. However, if the two graphs are not the same, that is sufficient evidence to determine it is not an identity. Just like paternity tests, you can rule out, but you can't rule in!

Practice Problems

1. $\sin^2 x = \dfrac{1-\cos 2x}{2}$ *Hint:* Remember to put (sin(x))^2
2. $\sin 3x = 3 \sin x$
3. $\cos x = 1/\tan x$

✔ Progress Check

Use identities to transform the expression

$$\tan t \cos t + \sin t + \frac{1}{\csc t}$$

into $3 \sin t$.

WARNING
You cannot verify an identity by checking to see that it "works" for one or more values of the variable as these values could turn out to be solutions to a conditional equation. (See Section 2.1.) You must show that an equation is true for *all* values in the domain of its variable to prove that it is an identity.

Exercise Set 7.4

In Exercises 1–12, determine the reference angle for each angle.

1. $250°$
2. $-130°$
3. $-330°$
4. $125°$
5. $\frac{6\pi}{5}$
6. $\frac{9\pi}{5}$
7. $335°$
8. $-10°$
9. $-47°$
10. $110°$
11. $\frac{15\pi}{7}$
12. $-\frac{3\pi}{5}$

In Exercises 13–24, use the rectangular coordinates of $P(t)$ to find $\sin t$ and $\cos t$.

13. $t = \frac{5\pi}{3}$
14. $t = \frac{3\pi}{4}$
15. $t = -5\pi$
16. $t = -\frac{5\pi}{4}$
17. $t = \frac{7\pi}{4}$
18. $t = \frac{7\pi}{6}$
19. $t = \frac{2\pi}{3}$
20. $t = \frac{5\pi}{6}$
21. $t = -\frac{\pi}{3}$
22. $t = -\frac{5\pi}{6}$
23. $t = \frac{5\pi}{4}$
24. $t = -11\pi$

In Exercises 25–40, without using tables or a calculator, find the values of the six trigonometric functions for each argument.

25. $135°$
26. $300°$
27. $-30°$
28. $-300°$
29. $\frac{\pi}{3}$
30. $\frac{\pi}{6}$
31. $\frac{\pi}{4}$
32. $\frac{\pi}{2}$
33. $\frac{5\pi}{6}$
34. $\frac{4\pi}{3}$
35. $\frac{3\pi}{2}$
36. $\frac{7\pi}{4}$
37. $\frac{3\pi}{4}$
38. $-\frac{11\pi}{6}$
39. $-\frac{5\pi}{4}$
40. $-\frac{7\pi}{6}$

In Exercises 41–56, find the rectangular coordinates of the given point.

41. $P(5\pi)$
42. $P\left(\frac{5\pi}{2}\right)$
43. $P\left(-\frac{\pi}{4}\right)$
44. $P\left(-\frac{3\pi}{2}\right)$
45. $P\left(\frac{5\pi}{4}\right)$
46. $P(8\pi)$
47. $P\left(\frac{4\pi}{3}\right)$
48. $P\left(\frac{2\pi}{3}\right)$
49. $P\left(-\frac{2\pi}{3}\right)$
50. $P\left(-\frac{19\pi}{3}\right)$
51. $P\left(\frac{19\pi}{6}\right)$
52. $P\left(\frac{17\pi}{6}\right)$
53. $P\left(-\frac{5\pi}{6}\right)$
54. $P\left(-\frac{11\pi}{6}\right)$
55. $P\left(\frac{19\pi}{3}\right)$
56. $P\left(\frac{25\pi}{3}\right)$

In Exercises 57–64, determine both a positive and a negative real number t, $|t| < 2\pi$, for which $P(t)$ has the following rectangular coordinates.

57. $(-1, 0)$
58. $(0, -1)$
59. $\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
60. $\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

61. $\left(-\dfrac{\sqrt{3}}{2}, \dfrac{1}{2}\right)$

62. $\left(-\dfrac{1}{2}, -\dfrac{\sqrt{3}}{2}\right)$

63. $\left(\dfrac{1}{2}, -\dfrac{\sqrt{3}}{2}\right)$

64. $\left(-\dfrac{\sqrt{3}}{2}, -\dfrac{1}{2}\right)$

65. Given $P(t) = (\frac{3}{5}, \frac{4}{5})$ use the symmetries of the circle to find

(a) $P(t + \pi)$ (b) $P\left(t - \dfrac{\pi}{2}\right)$

(c) $P(-t)$ (d) $P(-t - \pi)$

66. Given $P(t) = (-\frac{4}{5}, -\frac{3}{5})$, use the symmetries of the circle to find

(a) $P(t - \pi)$ (b) $P\left(t + \dfrac{\pi}{2}\right)$

(c) $P(-t)$ (d) $P(-t + \pi)$

67. If the point (a, b) is on the unit circle, show that $(a, -b)$, $(-a, b)$, and $(-a, -b)$ also lie on the unit circle.

68. If the point (a, b) is on the unit circle, show that (b, a), $(b, -a)$, $(-b, a)$, and $(-b, -a)$ also lie on the unit circle.

In Exercises 69–82, find the quadrant in which $P(t)$ lies if the following conditions hold.

69. $\sin t > 0, \cos t < 0$

70. $\sin t < 0, \tan t > 0$

71. $\cos t < 0, \tan t > 0$

72. $\tan t < 0, \sin t > 0$

73. $\sin t < 0, \cos t < 0$

74. $\tan t < 0, \cos t < 0$

75. $\sec t < 0, \sin t < 0$

76. $\tan t < 0, \sec t < 0$

77. $\csc t > 0, \sec t < 0$

78. $\sin t < 0, \cot t > 0$

79. $\sec t < 0, \cot t > 0$

80. $\cot t < 0, \sin t > 0$

81. $\sec t < 0, \csc t < 0$

82. $\csc t < 0, \cot t > 0$

In Exercises 83–94, find the value of the trigonometric function when t is replaced by $-t$. (For example, given $\sin t = \frac{1}{2}$, find $\sin(-t)$).

83. $\tan t = \dfrac{3}{2}$

84. $\sin t = 1$

85. $\tan t = 1$

86. $\cos t = -1$

87. $\tan t = \dfrac{\sqrt{2}}{2}$

88. $\cos t = \dfrac{\sqrt{3}}{2}$

89. $\cos t = -\dfrac{\sqrt{3}}{2}$

90. $\sin t = -\dfrac{1}{2}$

91. $\tan t = \sqrt{3}$

92. $\sin t = \dfrac{1}{2}$

93. $\sin t = \dfrac{\sqrt{3}}{2}$

94. $\tan t = \dfrac{\sqrt{3}}{3}$

95. If the point $(\frac{12}{13}, \frac{5}{13})$on the unit circle corresponds to the real number t, use the symmetries of the circle to find the coordinates of the point on the unit circle corresponding to the real number

(a) $t + \pi$ (b) $t - \dfrac{\pi}{2}$

(c) $-t$ (d) $-t - \pi$

96. If the point $(-\frac{4}{5}, -\frac{3}{5})$ on the unit circle corresponds to the real number t, use the symmetries of the circle to find the coordinates of the point on the unit circle corresponding to the real number

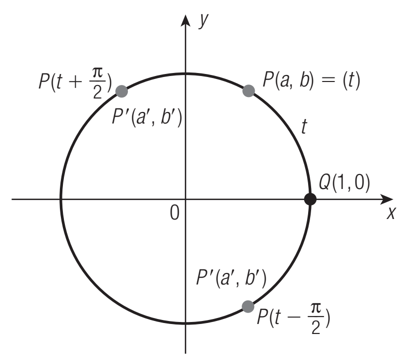
(a) $t - \pi$ (b) $t + \dfrac{\pi}{2}$

(c) $-t$ (d) $-t + \pi$

97. The point $P(a, b)$ on the unit circle corresponding to the real number t lies in quadrant II. Find the values of

(a) $\sin\left(t + \dfrac{\pi}{2}\right)$ (b) $\cos\left(t + \dfrac{\pi}{2}\right)$

In Exercises 98–100, the points $P(t) = P(a, b)$ and $P(t \pm \frac{\pi}{2}) = P'(a', b')$, as shown in the accompanying figure.



98. Show that $\frac{b'}{a'} = -\frac{a}{b}$. (*Hint:* Show that the lines OP and OP' are perpendicular and then determine their slopes.)

99. Show that $b' = \pm a$ and $a' = \pm b$. (*Hint:* The radii OP and OP' are equal in length. Use the distance formula combined with the result of Exercise 98 above to substitute alternately for b' and for a'.)

100. Show that either (i) $(a', b') = (-b, a)$ or (ii) $(a', b') = (b, -a)$. (*Hint:* Begin with the result of Exercise 99 and apply the result of Exercise 98.)

In Exercises 101–108, find the indicated value under the given conditions.

101. If $\sin t = \frac{3}{5}$ and t is in quadrant II, find $\tan t$.

102. If $\tan t = -\frac{3}{4}$ and t is in quadrant II, find $\cos t$.

103. If $\cos t = -\frac{5}{13}$ and t is in quadrant III, find $\sin t$.

104. If $\sin t = -\frac{5}{13}$ and t is in quadrant III, find $\tan t$.

105. If $\cos t = \frac{4}{5}$ and $\sin t < 0$, find $\sin t$.

106. If $\tan t = \frac{12}{5}$ and $\cos t < 0$, find $\sin t$.

107. If $\sin t = -\frac{3}{5}$ and $\tan t < 0$, find $\cos t$.

108. If $\tan t = -\frac{5}{12}$ and $\sin t > 0$, find $\sin t$.

In Exercises 109–120, find the values of the trigonometric functions of the angle θ if the point P lies on the terminal side of θ.

109. $P(-5, 12)$ 110. $P(3, -4)$

111. $P(-1, -1)$ 112. $P(1, 2)$

113. $P(-8, 6)$ 114. $P(12, 5)$

115. $P(12, -5)$ 116. $P(-1, \sqrt{3})$

117. $P(-12, -5)$ 118. $P(-3, 4)$

119. $P(-2, 1)$ 120. $P(-2, -1)$

In Exercises 121–126, use a calculator and the polynomial approximations

$$\sin t \approx t - \frac{t^3}{6} + \frac{t^5}{120} - \frac{t^7}{5040}$$

$$\cos t \approx 1 - \frac{t^2}{2} + \frac{t^4}{24} - \frac{t^6}{720}$$

to test the accuracy of the approximation.

121. $\sin 0.80$ 122. $\cos 1.10$

123. $\sin(-0.20)$ 124. $\cos(-0.75)$

125. $\tan 0.1$ 126. $\tan(-1.2)$

127. Using the polynomial approximation for $\sin t$ given above, show that sine is an odd function, that is,
$\sin(-t) = -\sin(t)$.

128. Using the polynomial approximation for $\cos t$ given above, show that cosine is an even function, that is, $\cos(-t) = \cos(t)$.

In Exercises 129–138, use trigonometric identities to transform the first expression into the second.

129. $\tan t \cos t, \quad \sin t$

130. $\dfrac{\cos t}{\sin t}, \quad \dfrac{1}{\tan t}$

131. $\dfrac{1-\sin^2 t}{\sin t}, \quad \dfrac{\cos t}{\tan t}$

132. $\tan t \sin t + \cos t, \quad \dfrac{1}{\cos t}$

133. $\cos t\left(\dfrac{1}{\cos t} - \cos t\right), \quad \sin^2 t$

134. $\dfrac{1-\cos^2 t}{\sin t}, \quad \sin t$

135. $\dfrac{1-\cos^2 t}{\cos^2 t}, \quad \tan^2 t$

136. $\dfrac{\cos^2 t}{1-\sin t}, \quad 1 + \sin t$

137. $(\sin t - \cos t)^2, \quad 1 - 2\sin t\cos t$

138. $\dfrac{1}{1-\sin t} + \dfrac{1}{1+\sin t}, \quad \dfrac{2}{\cos^2 t}$

7.5 Graphs of the Trigonometric Functions

Periodic Functions

In Section 7.3, we observed a certain repetitive nature to the trigonometric functions. This repetition gives us an additional method for graphing these functions. Thus, once we know the structure of the pattern that repeats itself we may use that knowledge to draw the graph over any specified region. This characteristic makes these functions especially useful in describing cyclic phenomena in a wide variety of applications.

A function f is **periodic** if there exists a positive number c, such that

$$f(x + c) = f(x)$$

for all x in the domain of f. The least number $c > 0$ for which f is periodic is called the **period** of f.

Recall from Section 7.3,

For every integer n,

$$P(t + 2\pi n) = P(t)$$
$$\sin (t + 2\pi n) = \sin t$$
$$\cos (t + 2\pi n) = \cos t$$

In particular, for $n = 1$,

$$\sin (t + 2\pi) = \sin t \qquad \text{and} \qquad \cos (t + 2\pi) = \cos t$$

Therefore, $\sin t$ and $\cos t$ are periodic functions. Furthermore, they both have period 2π. (See Exercises 49 and 50.) Because of the definitions of $\tan t$, $\cot t$, $\sec t$, and $\csc t$ in terms of $\sin t$ and $\cos t$, we note that these four trigonometric functions are also periodic. However, as we will see, not all of them have period 2π.

Graphs of Sine and Cosine

The definitions of $\sin t$ and $\cos t$ as the coordinates of points on the unit circle $x^2 + y^2 = 1$ tell us that they must satisfy the inequalities

$$-1 \le \sin t \le 1$$
$$-1 \le \cos t \le 1$$

Therefore, we indicate these minimum and maximum values on the y-axis as shown in Figure 46. Since these functions have a period of 2π, we initially sketch their graphs over the interval $[0, 2\pi]$.

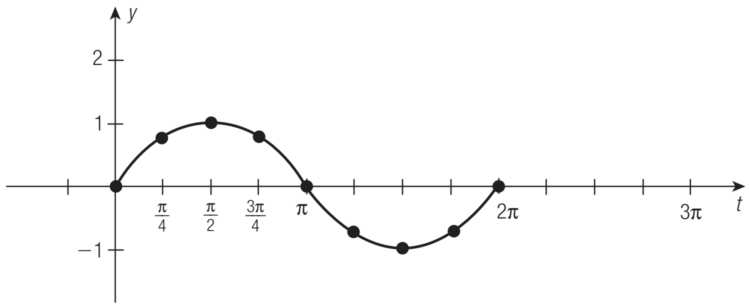


FIGURE 46 Graph of $y = \sin t$ for $0 \le t \le 2\pi$

We form Table 8 using the values from Table 7 to two decimal places. Since we are letting t denote the independent variable, we use the label t for the horizontal axis. Verify the values in Table 8 using your calculator.

TABLE 8 Plotting sin t and cos t

t	$\sin t$	$\cos t$
0	0	1
$\frac{\pi}{4}$	0.71	0.71
$\frac{\pi}{2}$	1	0
$\frac{3\pi}{4}$	0.71	−0.71
π	0	−1
$\frac{5\pi}{4}$	−0.71	−0.71
$\frac{3\pi}{2}$	−1	0
$\frac{7\pi}{4}$	−0.71	0.71
2π	0	1

In Figure 47, we repeat the graph of Figure 46 for adjacent intervals of width 2π.

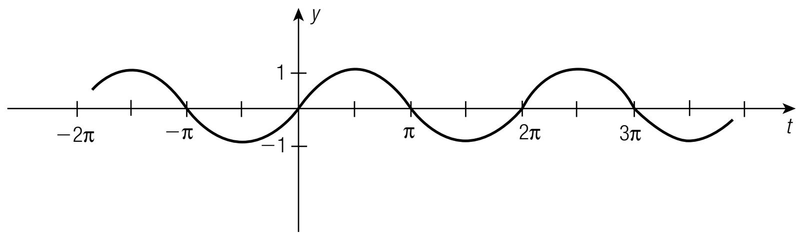


FIGURE 47 Graph of $y = \sin t$

We sketch $y = \cos t$ in Figure 48 using the values from Table 8 and the same procedure that we used to graph $y = \sin t$.

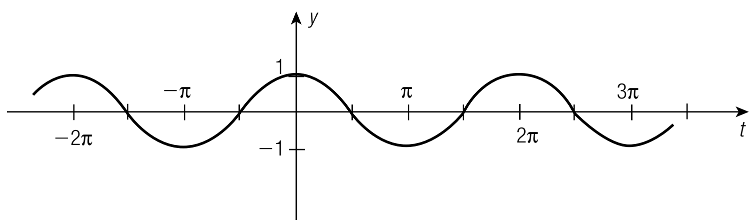


FIGURE 48 Graph of $y = \cos t$

Graphs of Tangent, Cotangent, Secant and Cosecant

Since $\sin t$ and $\cos t$ both have period 2π, one might suspect that $\tan t$ has period 2π as well. However, consider Figure 49. In Figure 49(a),

$$\tan(t + \pi) = \frac{\sin(t + \pi)}{\cos(t + \pi)} = \frac{-y}{-x} = \frac{y}{x} = \frac{\sin t}{\cos t} = \tan t$$

whereas in Figure 49(b),

$$\tan(t + \pi) = \frac{\sin(t + \pi)}{\cos(t + \pi)} = \frac{-y}{x} = \frac{y}{-x} = \frac{\sin t}{\cos t} = \tan t$$

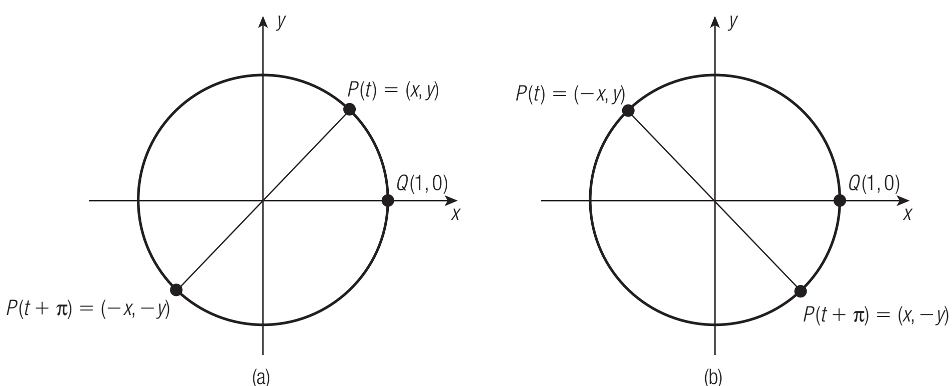


FIGURE 49 Circle Symmetry

Thus,

$$\tan(t + \pi) = \tan t$$

Since there are no other numbers c with $0 < c < \pi$ such that $\tan(t + c) = \tan t$ (See Exercise 51.), $\tan t$ has period π.

It turns out to be convenient to plot $\tan t$ in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$, which is of length π. First, we form Table 9 using special values of t as well as a calculator to two decimal places. Since $\tan t$ is undefined at $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ we need to carefully consider the behavior of the graph *near* these values of t. That is why Table 9 contains the values -1.5 and -1.57 near $-\frac{\pi}{2}$, and 1.5 and 1.57 near $\frac{\pi}{2}$. As t gets closer and closer to $\frac{\pi}{2}$ with $t < \frac{\pi}{2}$, we say that $\tan t$ **approaches infinity**. Similarly, as t gets closer and closer to $-\frac{\pi}{2}$ with $t > -\frac{\pi}{2}$, we say that $\tan t$ **approaches negative infinity**. These considerations lead us to the graph of $\tan t$ as shown in Figure 50. Note the asymptotes in this graph, two of which are found at $t = \pm\frac{\pi}{2}$.

TABLE 9 Plotting $\tan t$

t	$\tan t$
$-\frac{\pi}{2}$	undefined
-1.57	-1255.77
-1.5	-14.10
$-\frac{\pi}{3}$	-1.73
$-\frac{\pi}{4}$	-1
$-\frac{\pi}{6}$	-0.58
0	0
$\frac{\pi}{6}$	0.58
$\frac{\pi}{4}$	1
$\frac{\pi}{3}$	1.73
1.5	14.10
1.57	1255.77
$\frac{\pi}{2}$	undefined

We may now use the graphs of $\sin t$, $\cos t$, and $\tan t$ to assist us in sketching the graphs of $\csc t$, $\sec t$, and $\cot t$, respectively. Since $\csc t = \frac{1}{\sin t}$, we compute the reciprocal of the y-coordinate of $\sin t$ at a point to determine the y-coordinate of $\csc t$ at that point. Note that we cannot form the reciprocal if $\sin t = 0$, that is, if $t = n\pi$ for any integer n. Thus, $\csc t$ is undefined for $t = n\pi$ just as $\tan t$ is undefined if $t = \frac{\pi}{2} + n\pi$.

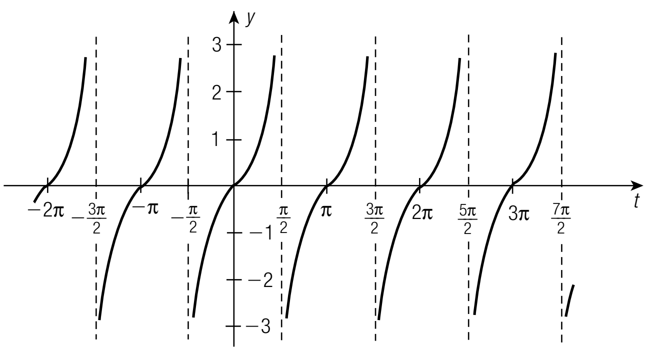


FIGURE 50 Graph of $y = \tan t$

In Figure 51, we sketch $y = \csc t$ with its reciprocal $y = \sin t$ as a reference.

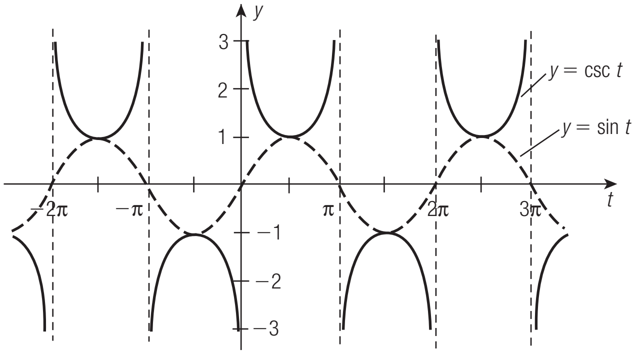


FIGURE 51 Graph of $y = \csc t$

Similarly in Figure 52, we sketch $y = \sec t$ with reference $y = \cos t$, and lastly, in Figure 53, we sketch $y = \cot t$ with its reference $y = \tan t$.

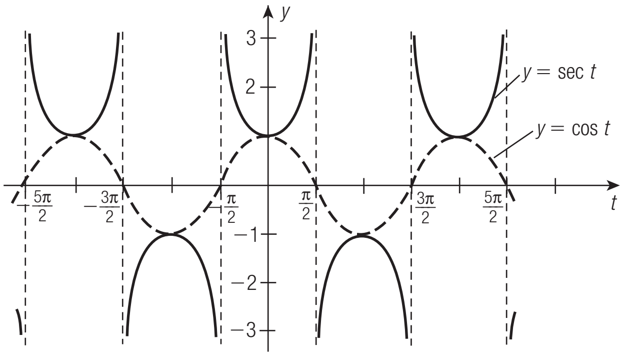


FIGURE 52 Graph of $y = \sec t$

We summarize the period, domain, and range for the six trigonometric functions, as shown in Table 10.

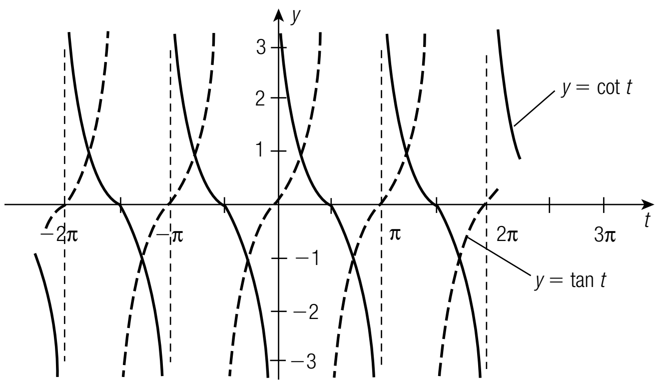


FIGURE 53 Graph of $y = \cot t$

TABLE 10 Domain and Range for Trigonometric Functions

Function	Period	Domain	Range
$\sin t$	2π	all t	$[-1, 1]$
$\cos t$	2π	all t	$[-1, 1]$
$\tan t$	π	$t \neq \frac{\pi}{2} + n\pi$	$(-\infty, \infty)$
$\csc t$	2π	$t \neq n\pi$	$(-\infty, -1], [1, \infty)$
$\sec t$	2π	$t \neq \frac{\pi}{2} + n\pi$	$(-\infty, -1], [1, \infty)$
$\cot t$	π	$t \neq n\pi$	$(-\infty, \infty)$

EXAMPLE 1 GRAPHING BY "ADDITION"

Sketch the graph of $f(t) = 1 + \sin t$.

SOLUTION

Rather than form a table of values and plot points, we note that the graph of $f(t) = 1 + \sin t$ is that of the graph of $f(t) = \sin t$ shifted up one unit, as shown in Figure 54. (See Section 3.3.)

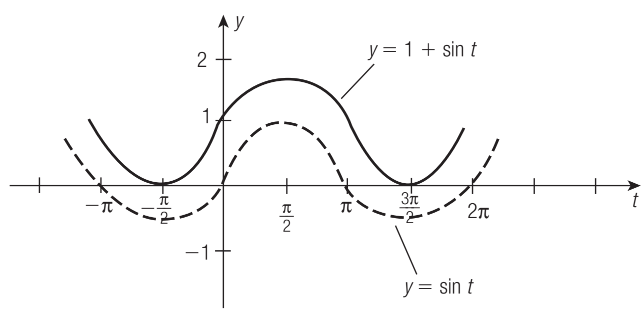


FIGURE 54 Graph of $y + \sin t$

EXAMPLE 2 GRAPHING BY "ADDITION"

Sketch the graph of $f(t) = \sin t + \cos t$.

SOLUTION

We note that the y-coordinate of $f(t) = \sin t + \cos t$ is the sum of the y coordinates of $\sin t$ and $\cos t$ for each value of t. We do this geometrically for each value of t as shown in Figure 55. In Figure 56, we present the final result.

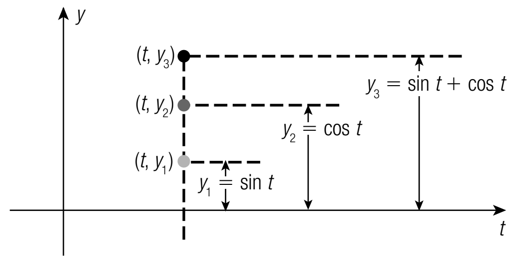


FIGURE 55 Adding y-coordinates Geometrically

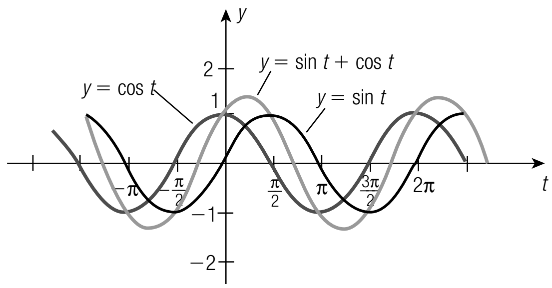


FIGURE 56 Graph of $y = \sin t + \cos t$

Graphing Calculator Power User's Corner 7.5

When graphing trigonometric functions, your calculator *must* be set to RADIAN mode. (*Note:* If your calculator has built-in trigonometric RANGE settings and your calculator is set to degree mode, you will get what appears to be a correct graph. However, using TRACE will reveal incorrect function values.)

For each graph in this section, set the RANGE values on your calculator to match those in the figure and GRAPH the function or functions shown. Repeat this activity for the graphs in the next section.

Exercise Set 7.5

In Exercises 1–38, use the graph of the particular trigonometric function to find the values of t in the interval $[0, 2\pi)$ that satisfy the given equation.

1. $\cos t = 0$
2. $\sin t = 1$
3. $\tan t = 1$
4. $\cos t = -1$
5. $\sin t = \frac{\sqrt{2}}{2}$
6. $\cos t = \frac{\sqrt{3}}{2}$
7. $\cos t = -\frac{\sqrt{3}}{2}$
8. $\sin t = -1$
9. $\tan t = \sqrt{3}$
10. $\sin t = \frac{3}{2}$
11. $\sin t = \frac{\sqrt{3}}{2}$
12. $\tan t = \frac{\sqrt{3}}{3}$
13. $\sin t = -\frac{1}{2}$
14. $\cos t = -\frac{\sqrt{2}}{2}$

15. $\sin t = 2$

16. $\cos t = \dfrac{\sqrt{2}}{2}$

17. $\sin t = \dfrac{1}{2}$

18. $\cos t = \dfrac{3}{2}$

19. $\sec t = 1$

20. $\sec t = -1$

21. $\csc t = -2$

22. $\csc t = 0$

23. $\cot t = 1$

24. $\cot t = \sqrt{3}$

25. $\cot t = -1$

26. $\cot t = \dfrac{\sqrt{3}}{3}$

27. $\sec t = \sqrt{2}$

28. $\csc t = -\sqrt{2}$

29. $\cot t = -\sqrt{3}$

30. $\csc t = \dfrac{2\sqrt{3}}{3}$

31. $\sin t = \dfrac{1}{2}, \quad \sec t < 0$

32. $\tan t = \sqrt{3}, \quad \csc t < 0$

33. $\sec t = -2, \quad \csc t > 0$

34. $\csc t = -2, \quad \cot t > 0$

35. $\csc t = -\sqrt{2}, \quad \sec t < 0$

36. $\sec t = \sqrt{2}, \cot t > 0$

37. $\cot t = -1, \quad \sec t < 0$

38. $\cot t = \sqrt{3}, \quad \csc t < 0$

In Exercises 39–48, sketch the graph of the given function. Then, determine appropriate RANGE values and GRAPH the functions on your graphing calculator. Be sure your calculator is in RADIAN mode. Replace the variable t with the variable x.

39. $f(t) = 1 + \cos t$

40. $f(t) = -1 + \sin t$

41. $f(t) = 2 \sin t$

42. $f(t) = \dfrac{1}{2} \cos t$

43. $f(t) = \sin t + \dfrac{1}{2} \cos t$

44. $f(t) = 2 \sin t + \cos t$

45. $f(t) = \sin t - \cos t$

46. $f(t) = \sin(-t) + \cos t$

47. $f(t) = t + \sin t$

48. $f(t) = -t + \cos t$

49. Prove that the period of the sine function is 2π. (*Hint:* Assume $\sin(t + c) = \sin t$, $0 < c < 2\pi$, for all t. By letting $t = 0$, show that $\sin c = 0$ and, consequently, that $c = \pi$. Finally, conclude that $\sin(t + \pi) = \sin t$ does not hold for $t = \frac{\pi}{2}$.)

50. Prove that the period of the cosine function is 2π.

51. Prove that the period of the tangent function is π.

52. Verify that $\sin(-t) = -\sin t$ by using the graph of the sine function.

53. Verify that $\cos(-t) = \cos t$ by using the graph of the cosine function.

54. Determine the domain and range of the functions in Exercises 39–48 by examining the graph of each function.

55. Use the identity
$$\tan t = \frac{\sin t}{\cos t}$$
to determine the vertical asymptotes of the graph of $\tan t$.

7.6 Graphs: Amplitude, Period, and Phase Shift

Our objective in this section is to sketch graphs, such as $f(x) = A \sin(Bx + C)$ and $f(x) = A \cos(Bx + C)$, where A, B, and C are real numbers with $A \neq 0$ and $B \neq 0$. (We suggest reviewing Section 3.3 at this time.) Note that we now use the symbol x to denote the independent variable rather than the symbol t, which was used in the previous sections of this chapter. The use of the independent variable x in this context should not be confused with the x-coordinate of the point on the unit circle, $P(t) = P(x, y)$.

Amplitude

Since the sine function and the cosine function both have a maximum value of 1 and a minimum value of -1, the functions $f(x) = A \sin x$ and $f(x) = A \cos x$ have a maximum value of $|A|$ and a minimum value of $-|A|$. We define

> The **amplitude** of $f(x) = A \sin x$ or $f(x) = A \cos x$ is $|A|$.

The multiplier A acts as a vertical "stretching" factor when $|A| > 1$, and as a vertical "shrinking" factor when $|A| < 1$.

EXAMPLE 1 AMPLITUDE OF A SIN *X*

Sketch the graphs of $y = 2 \sin x$ and $y = \frac{1}{2}\sin x$ on the same coordinate axes.

SOLUTION

The graph of $y = 2 \sin x$ has an amplitude of 2. Its maximum value is 2 and its minimum value is -2. Similarly, the graph of $y = \frac{1}{2}\sin x$ has an amplitude of $\frac{1}{2}$ with maximum and minimum values of $\frac{1}{2}$ and $-\frac{1}{2}$, respectively. Both graphs are shown in Figure 57.

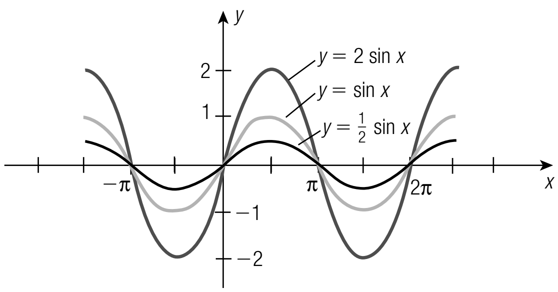


FIGURE 57 Graphs of $y = 2 \sin x$ and $y = {}^1/_2 \sin x$

EXAMPLE 2 GRAPHING WITH A NEGATIVE "STRETCHING" FACTOR

Sketch the graph of $f(x) = -3 \cos x$.

SOLUTION

The graph of $y = -3 \cos x$ has an amplitude of 3 with maximum and minimum values of 3 and -3, respectively. This graph, as shown in Figure 58, is said to be a **reflection** about the x-axis of the graph $y = 3 \cos x$.

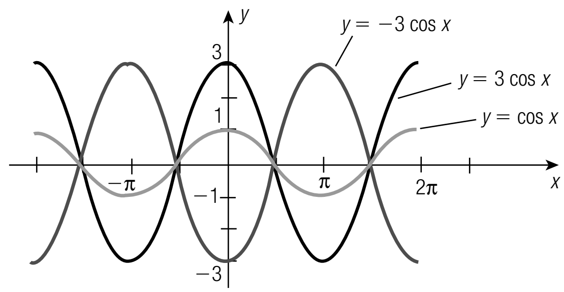


FIGURE 58 Graph of $y = -3 \cos x$

Period

We now seek to determine the period of functions such as $f(x) = A \sin Bx$ or $f(x) = A \cos Bx$. Although the argument of the functions has been generalized from x to Bx from our earlier presentation, the amplitude remains unchanged, that is, the amplitude of either function is $|A|$. Now, since $y = \sin x$ and $y = \cos x$ both have period 2π, we say that either function completes one **cycle** as x varies from 0 to 2π. Similarly, $f(x) = A \sin Bx$ or $f(x) = A \cos Bx$ completes one cycle as Bx varies from 0 to 2π if $B > 0$, or as $-Bx$ varies from 0 to 2π if $B < 0$. We may combine both cases by writing, "as $|B|x$ varies from 0 to 2π." This leads to the equation

$$|B|x = 0 \qquad \text{in which case} \qquad x = 0$$

and to the equation

$$|B|x = 2\pi \qquad \text{in which case} \qquad x = \frac{2\pi}{|B|}$$

Therefore, both $f(x) = A \sin Bx$ and $f(x) = A \cos Bx$ complete one cycle as x varies from 0 to $\frac{2\pi}{|B|}$, or equivalently,

> The **period** of $f(x) = A \sin Bx$ or $f(x) = A \cos Bx$ is $\dfrac{2\pi}{|B|}$.

The multiplier B acts as a horizontal stretching factor if $0 < |B| < 1$ and as a horizontal shrinking factor if $|B| > 1$.

EXAMPLE 3 THE PERIOD OF SIN *Bx*

Sketch the graph of $f(x) = \sin 2x$.

SOLUTION

Since $B = 2$, the period is $\frac{2\pi}{2} = \pi$. Therefore, the graph completes a cycle every π units, as shown in Figure 59.

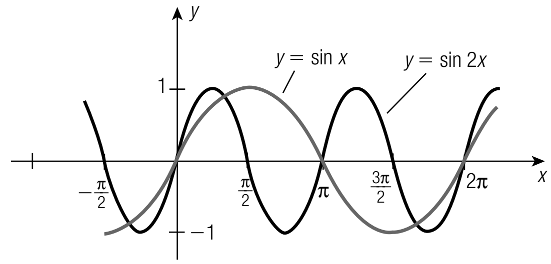


FIGURE 59 Graph of $y = \sin 2x$

EXAMPLE 4 GRAPHING *A* COS *Bx*

Sketch the graph of $f(x) = 2 \cos \frac{1}{2}x$.

SOLUTION

Since $B = \frac{1}{2}$, the period is $2\pi/\frac{1}{2} = 4\pi$. The graph completes a cycle every 4π units. Note that the amplitude is 2, which provides us with maximum and minimum values of 2 and -2, respectively. The graph is shown in Figure 60.

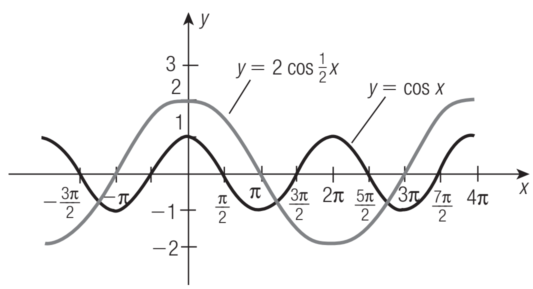


FIGURE 60 Graph of $y = 2 \cos \frac{1}{2}x$

Phase Shift

We now examine the more general functions $f(x) = A \sin (Bx + C)$ and $f(x) = A \cos (Bx + C)$. As before, generalizing the argument of the trigonometric function does not affect the amplitude, which is $|A|$. Since $y = \sin x$ and $y = \cos x$ both complete one cycle as x varies from 0 to 2π, this leads to the equation

$$Bx + C = 0 \qquad \text{in which case} \qquad x = -\frac{C}{B}$$

and to the equation

$$Bx + C = 2\pi \qquad \text{in which case} \qquad x = \frac{2\pi - C}{B}$$

The value $-\frac{C}{B}$ is called the **phase shift**. Consider the function

$$f(x) = A \sin (Bx + C).$$

If $-\frac{C}{B} > 0$, shifting the graph of $y = A \sin Bx$ to the right $-\frac{C}{B}$ units yields the graph of $y = A \sin (Bx + C)$. If $-\frac{C}{B} < 0$, shifting the graph of $y = A \sin Bx$ to the left $\frac{C}{B}$ units yields the graph of $y = A \sin (Bx + C)$. Similar statements can be made about $y = A \cos Bx$ and $y = A \cos (Bx + C)$.

Consider the functions $f(x) = A \sin (Bx + C)$ or $f(x) = A \cos (Bx + C)$ for $A \neq 0$, $B \neq 0$.

(a) The amplitude of $f(x)$ is $|A|$.

(b) The period of $f(x)$ is $\dfrac{2\pi}{|B|}$

(c) The phase shift of $f(x)$ is $-\dfrac{C}{B}$.

EXAMPLE 5 GRAPHING WITH A PHASE SHIFT

Sketch the graph $f(x) = 3 \sin (2x - \pi)$.

SOLUTION

Graphing $f(x) = A \sin (Bx + C)$

Step 1. Determine A, B, and C.

Step 1. Since

$$f(x) = 3 \sin (2x - \pi) = A \sin (Bx + C)$$
$$A = 3,\ B = 2,\text{ and } C = -\pi$$

Step 2. Determine the amplitude, period, and phase shift.

Step 2.

$$\text{amplitude} = |A| = 3$$

$$\text{period} = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi$$

$$\text{phase shift} = -\frac{C}{B} = \frac{\pi}{2}$$

or, $2x - \pi = 0$ yields $x = \frac{\pi}{2}$ as the phase shift.

Step 3. Analyze the effect of the phase shift on the point (0, 0).

Step 3. A phase shift of $\frac{\pi}{2}$ causes the cycle to "begin" at $(\frac{\pi}{2}, 0)$ rather than at (0, 0).

Step 4. Use the period to determine the values of x at which a cycle is complete.

Step 4. Adding the period π to the phase shift $\frac{\pi}{2}$, we have

$$x = \frac{\pi}{2} + \pi = \frac{3\pi}{2}$$

The graph completes a cycle in the interval

$$\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$$

Step 5. Usng the amplitude, sketch the graph.

Step 5. Recalling that the amplitude is 3, see the graph sketched in Figure 61.

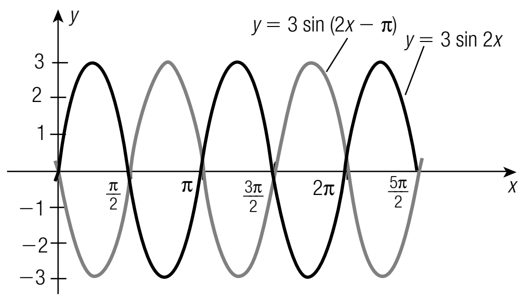
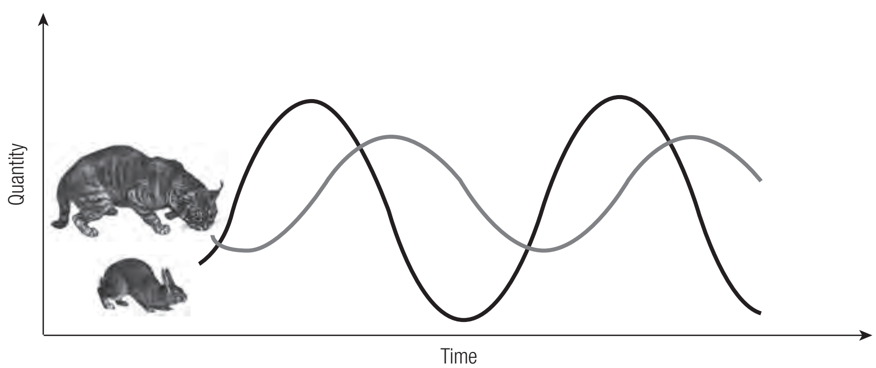


FIGURE 61 Graph of $y = 3 \sin (2x - \pi)$

Focus on Predator–Prey Interaction

In the natural world, we frequently find that two plant or animal species interact in their environment in such manner that one species, the prey, serves as the primary food supply for the second species, the predator. Examples of such interaction are the relationships between trees (prey) and insects (predators) and between rabbits (prey) and lynxes (predators). As the population of the prey increases, the additional food supply results in an increase in the population of the predators. More predators consume more food, so the population of the prey decreases, which, in turn, leads, to a decrease in the population of the predators. The reduction in the predator population results in an increase in the number of prey and the cycle starts all over again.

The accompanying figure, adapted from *Mathematics: Ideas and Applications*, by Daniel D. Benice, Academic Press, 1978 (used with permission), shows the interaction between lynx and rabbit populations. Both curves demonstrate periodic behavior and can be described by trigonometric functions.



✔ Progress Check

If $f(x) = 2\cos\left(2x + \frac{\pi}{2}\right)$, find the amplitude, period, and phase shift of f. Sketch the graph of the function.

Answers

The amplitude = 2, the period = π, and the phase shift = $-\frac{\pi}{4}$. See Figure 62.

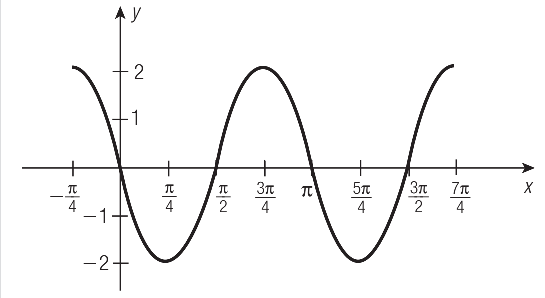


FIGURE 62 Graph of $y = 2\cos(2x + \pi/2)$

Exercise Set 7.6

In Exercises 1–12, determine the amplitude and period, and sketch the graph of each of the following functions. Then, determine appropriate RANGE values and GRAPH the functions on your graphing calculator. Be sure your calculator is in RADIAN mode.

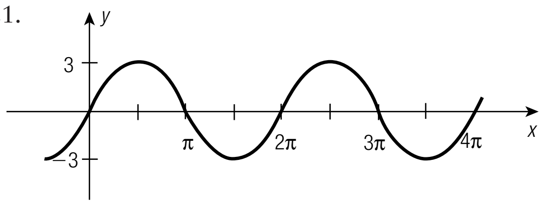
1. $f(x) = 3\sin x$
2. $f(x) = \frac{1}{4}\cos x$
3. $f(x) = \cos 4x$
4. $f(x) = \sin\frac{x}{4}$
5. $f(x) = -2\sin 4x$
6. $f(x) = -\cos\frac{x}{4}$
7. $f(x) = 2\cos\frac{x}{3}$
8. $f(x) = 4\sin 4x$
9. $f(x) = \frac{1}{4}\sin\frac{x}{4}$
10. $f(x) = \frac{1}{2}\cos\frac{x}{4}$
11. $f(x) = -3\cos 3x$
12. $f(x) = -2\sin 3x$

In Exercises 13–20, for each given function, determine the amplitude, period, and phase shift. Sketch the graph of the function. Then determine appropriate RANGE values and GRAPH the functions on your graphing calculator. Be sure your calculator is in RADIAN mode.

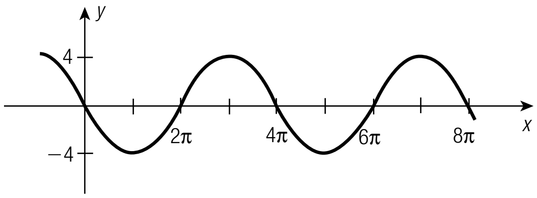
13. $f(x) = 2\sin(x - \pi)$
14. $f(x) = \frac{1}{2}\cos\left(x + \frac{\pi}{2}\right)$
15. $f(x) = 3\cos(2x - \pi)$
16. $f(x) = 4\sin\left(x + \frac{\pi}{4}\right)$
17. $f(x) = \frac{1}{3}\sin\left(3x + \frac{3\pi}{4}\right)$
18. $f(x) = 2\cos\left(2x + \frac{\pi}{2}\right)$
19. $f(x) = 2\cos\left(\frac{x}{4} - \pi\right)$
20. $f(x) = 6\sin\left(\frac{x}{2} + \frac{\pi}{2}\right)$

In Exercises 21–24, find a trigonometric function with the given graph. Verify your answer with your graphing calculator.

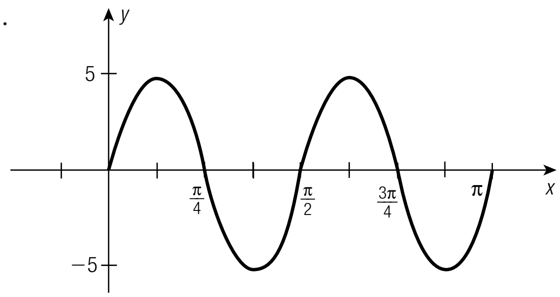
21.



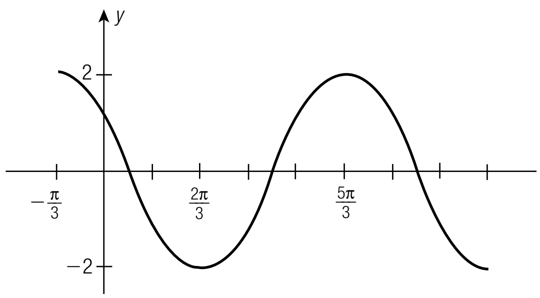
22.



23.

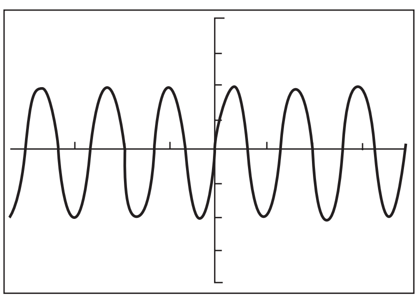


24.

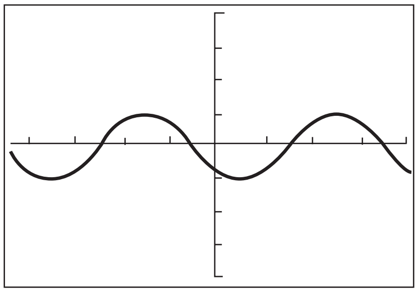


In Exercises 25–28, the graphs are variations of the function $y = \sin x$ or $y = \tan x$. Determine A, B, and C, where $y = A \sin (Bx + C)$ or $y = A \tan (Bx + C)$, for each graph below. Verify your answer by GRAPHing the function on your graphing calculator. (All graphs are drawn in the viewing rectangle $-6.28 \le X \le 6.28$, $-4 \le Y \le 4$, with XSCL = 1.57 and YSCL = 1.)

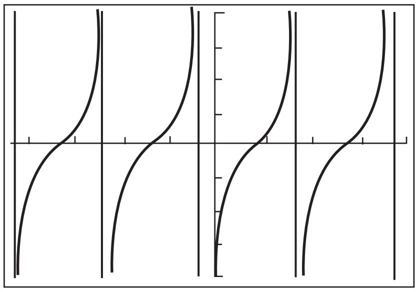
25.



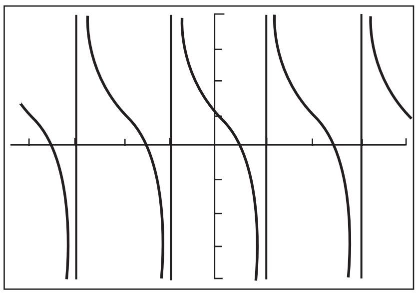
26.



27.



28.



29. GRAPH the following functions together in the TENS viewing rectangle to see the effect of *amplitude* on the graph of $y = \sin x$.

(a) $y = \sin x$ (b) $y = 3 \sin x$

(c) $y = 8 \sin x$ (d) $y = -5 \sin x$

30. GRAPH the following pairs of functions together in the viewing rectangle $-6.28 \le X \le 6.28$ and $-2 \le Y \le 2$ to see the effect of *period* on the graph of $y = \cos x$.

(a) $y = \cos x$ $y = \cos \frac{x}{2}$

(b) $y = \cos x$ $y = \cos 3x$

(c) $y = \cos x$ $y = \cos 5x$

31. GRAPH the following pairs of functions together in the viewing rectangle $-6.28 \le X \le 6.28$ and $-2 \le Y \le 2$ to see the effect of *period* on the graph of $y = \cos x$.

(a) $y = \cos x$ $\qquad y = \cos 75x$

(b) $y = \cos x$ $\qquad y = \cos 85x$

32. The graphs in Exercise 30 are "reasonable," but some of those in Exercise 31 are "not." TRACE the graphs of $y = \cos 75x$ and $y = \cos 85x$ to determine the points plotted by the calculator when graphing these functions. What points (x values) do you think it is necessary to plot to obtain a reasonable representation? Can you determine a viewing rectangle that gives "correct" representations of these graphs?

In Exercises 33–35, GRAPH the following pairs of functions together in the viewing rectangle $-6.28 \le X \le 6.28$ and $-2 \le Y \le 2$ to see the effect of *phase shift* on the graph of $y = \sin x$. TRACE to determine the x-intercept of each shifted graph.

33. $y = \sin x$ $\qquad y = \sin(x - \pi)$

34. $y = \sin x$ $\qquad y = \sin\left(x + \dfrac{\pi}{3}\right)$

35. $y = \sin 2x$ $\qquad y = \sin\left(2x + \dfrac{\pi}{3}\right)$

36. GRAPH the following functions together in the viewing rectangle $-6.28 \le X \le 6.28$ and $-2 \le Y \le 2$.

(a) $y = \sin x$ $\qquad$ (b) $y = x - \dfrac{x^3}{3!}$

(c) $y = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}$

(d) $y = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \dfrac{x^9}{9!} - \dfrac{x^{11}}{11!}$

37. The polynomials in Exercise 36 are called Taylor Polynomials for $\sin x$. Note that the graphs of the Taylor Polynomials with higher degree coincide with the graph of $y = \sin x$ over a larger interval of the x-axis. Can you determine the smallest Taylor Polynomial for $\sin x$ that coincides with $y = \sin x$ on the interval $[-2\pi, 2\pi]$?

38. Taylor Polynomials for $\cos x$ are

$$1 - \frac{x^2}{2!}$$

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

and so on.

Compare the graphs of these polynomials with the graph of $y = \cos x$.

7.7 The Inverse Trigonometric Functions

Inverse functions were introduced in Section 3.5 and also discussed in Section 6.1. Furthermore, they were used to define the logarithmic function in Section 6.3. We have seen that if f is a one-to-one function whose domain is the set X and whose range is the set Y, then the inverse function f^{-1} reverses the correspondence, that is,

$$f^{-1}(y) = x \qquad \text{for all } y \in y$$

if and only if

$$f(x) = y \qquad \text{for all } x \in x$$

Using this definition, we saw that the following identities characterize inverse functions.

$$f^{-1}[f(x)] = x \qquad \text{for all } x \text{ in } X$$
$$f[f^{-1}(y) = y \qquad \text{for all } y \text{ in } Y$$

If we attempt to find an inverse of the sine function, we have an immediate problem. Since sine is a periodic function, it is not a one-to-one function and has no inverse. However, we can resolve this problem by defining a function that agrees with the sine function, but over a restricted domain. That is, we would like to find an interval such that $y = \sin x$ is one-to-one and y assumes all values between -1 and 1 over this interval. If we define the function f by

$$f(x) = \sin x, \qquad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

then f assumes all real values in the interval $[-1, 1]$ as shown in Figure 63. We see that f is an increasing function and, therefore, one-to-one. Consequently, f has an inverse, and we are led to the following definition.

The inverse sine function, denoted by **arcsin**, or **sin^{-1}**, is defined by

$$\sin^{-1} x = y \qquad \text{if and only if} \qquad \sin y = x$$

where $-1 \leq x \leq 1$ and $-\dfrac{\pi}{2} \leq y \leq \dfrac{\pi}{2}$.

In words, the inverse sine of x takes on the value y, where the domain is $-1 \leq x \leq 1$ and the range is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. Sometimes $y = \sin^{-1} x$ is read, "y equals the arcsine of x" or "y is the angle whose sine is x."

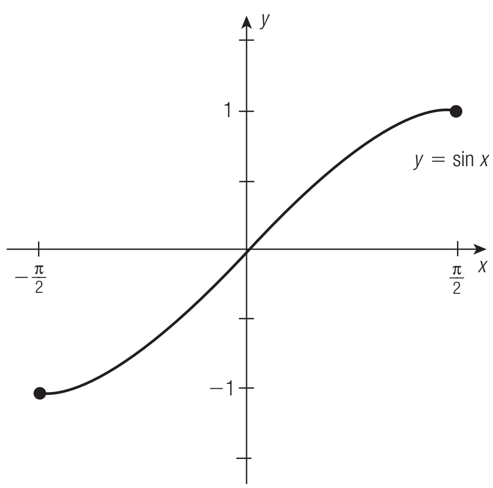


FIGURE 63 Graph of $y = \sin x$, $-{}^{\pi}/_{2} \leq x \leq {}^{\pi}/_{2}$

WARNING

When we defined $\sin^n t = (\sin t)^n$, we said that this definition does not hold when $n = -1$. This allowed us to reserve the notation $\sin^{-1}$ for the inverse sine function. Therefore, $\sin^{-1} x$ is not to be confused with

$$(\sin x)^{-1} = \frac{1}{\sin x}$$

The notation arcsin and $\sin^{-1}$ are both in common use, and we will use them interchangeably. (Recall from our presentation of the unit circle that the coordinates of $P(t)$ are (x, y), where $y = \sin t$. Thus, t determines an *arc whose sine is* y, that is, $t = \arcsin y$.)

We now form a table of values and sketch the graph of $y = \sin^{-1} x$, as shown in Figure 64. Observe that the restricted graph of $y = \sin x$ from Figure 63 and the graph of $y = \sin^{-1} x$ from Figure 64 are reflections of one another about the line $y = x$.

x	y
-1	$-\frac{\pi}{2}$
$-\frac{\sqrt{3}}{2}$	$-\frac{\pi}{3}$
$-\frac{1}{2}$	$-\frac{\pi}{6}$
0	0
$\frac{1}{2}$	$\frac{\pi}{6}$
$\frac{\sqrt{3}}{2}$	$\frac{\pi}{3}$
1	$\frac{\pi}{2}$

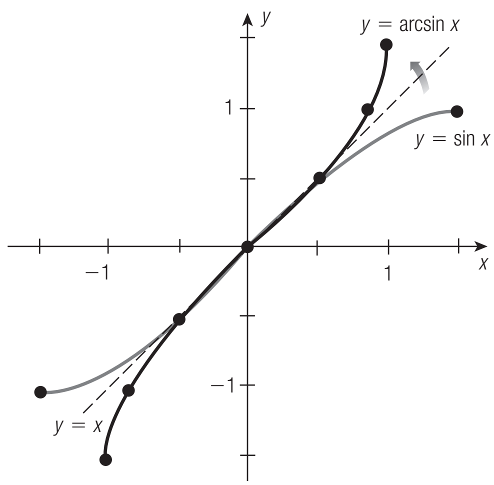


FIGURE 64 Graph of $y = \arcsin x$

EXAMPLE 1 APPLYING THE INVERSE TRIGONOMETRIC FUNCTIONS

Find.

(a) $\arcsin \frac{1}{2}$ (b) $\arcsin(-1)$

SOLUTION

(a) If $y = \arcsin \frac{1}{2}$, then $\sin y = \frac{1}{2}$, where y is restricted to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}.]$. Thus, $y = \frac{\pi}{6}$ is the *only* correct answer.

(b) If $y = \arcsin(-1)$, then $\sin y = -1$, where $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. Thus, $-\frac{\pi}{2}$ is the *only* correct answer.

EXAMPLE 2 APPLYING THE INVERSE TRIGONOMETRIC FUNCTIONS

Evaluate $\sin^{-1}\left(\cos \frac{\pi}{6}\right)$.

SOLUTION

Since $\cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$, we have

$$\sin^{-1}\left(\cos \frac{\pi}{6}\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

We let

$$y = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

Then

$$\sin y = \frac{\sqrt{3}}{2} \qquad \text{where} \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

and

$$y = \frac{\pi}{3}$$

is the *only* solution.

Graphing Calculator Power User's Corner 7.7

Graphing the Inverse Trig Functions

Any time a problem asks for the inverse or arc of a trig function, you can use [2nd] and then press the button for that function on the calculator. For example, when asked to find the value of arccos(tan $\frac{1}{20}$), you would type [2nd][cos]([tan](1/20)). The calculator would return the answer 1.520733709.

If we choose a value outside the domain of the particular inverse trig function, an error message is returned. For example, try entering [2nd][sin]([tan](5)). Why do you think this problem cannot be solved by the calculator? Could it be solved by hand?

The calculator can also be used to graph inverse trigonometric functions. Bear in mind that the arcsine, for example, can only be defined as a function when $-1 \le x \le 1$. So when graphing on the calculator, we type in Y_1=[2nd][sin](x). Then type [ZOOM] 7 to set the window to your standard trigonometric viewing window, and you will see that you get a graph with limited domain and range.

Since arcsine is the inverse operation of sine, what do you suppose the calculator would return as a graph of the arcsine of the sine function? Type Y_1 = [2nd][sin](sin(x)) and graph in the trig window. What do you see? The graph of this composition returns the identity line, $y = x$ for the domain of the sine inverse function from -1 to 1. Return to the home screen and try the following:

1. $\cos^{-1}(\cos 3)$
2. $\tan(\tan^{-1} 5)$
3. $\csc(\csc^{-1} 0.5)$

✔ Progress Check

Find.

(a) $\sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$ (b) $\arcsin\left(\tan\dfrac{5\pi}{4}\right)$

Answers

(a) $-\dfrac{\pi}{3}$ (b) $\dfrac{\pi}{2}$

We may use a similar approach to define the inverse cosine function. If we define the function f by

$$f(x) = \cos x, \qquad 0 \le x \le \pi$$

then f assumes all real values in the interval $[-1, 1]$ as shown in Figure 65. We see that f is a decreasing function and, hence, one-to-one. Therefore, f has an inverse, and we have the following definition.

The inverse cosine function, denoted by **arccos**, or **cos**$^{-1}$, is defined by

$$\cos^{-1} x = y \qquad \text{if and only if} \qquad \cos y = x$$

where $-1 \le x \le 1$ and $0 \le y \le \pi$.

We form a table of values and sketch the graph of $y = \cos^{-1} x$, as shown in Figure 66. Note that the graphs in Figures 65 and 66 are reflections of each other about the line $y = x$.

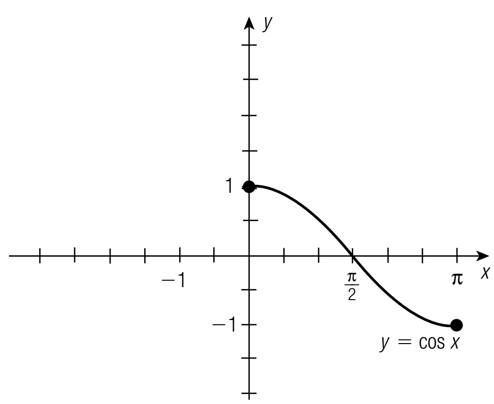


FIGURE 65 Graph of $y = \cos x$, $0 \le x \le \pi$

x	y
1	0
$\frac{\sqrt{3}}{2}$	$\frac{\pi}{6}$
$\frac{1}{2}$	$\frac{\pi}{3}$
0	$\frac{\pi}{2}$
$-\frac{1}{2}$	$\frac{2\pi}{3}$
$-\frac{\sqrt{3}}{2}$	$\frac{5\pi}{6}$
-1	π

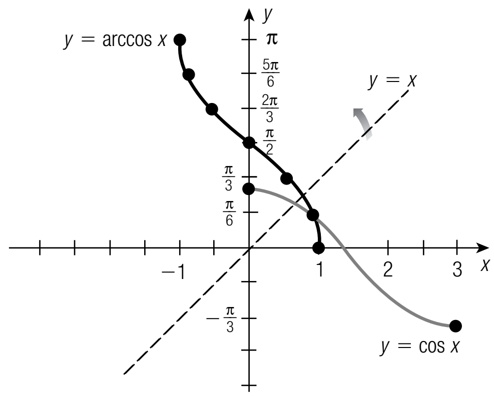


FIGURE 66 Graph of $y = \arccos x$

EXAMPLE 3 APPLYING THE INVERSE TRIGONOMETRIC FUNCTIONS

Find.

(a) $\cos^{-1}\left(-\frac{1}{2}\right)$ (b) $\arccos\left(\sin\frac{\pi}{2}\right)$

SOLUTION

(a) If $y = \cos^{-1}(-\frac{1}{2})$, then $\cos y = -\frac{1}{2}$, where y is restricted to the interval $[0, \pi]$. Consequently, $y = \frac{2\pi}{3}$ is the *only* correct answer.

(b) Since $\sin\frac{\pi}{2} = 1$, we let $y = \arccos 1$. Then $\cos y = 1$, where $0 \le y \le \pi$. Therefore, $y = 0$ is the *only* correct answer.

We define the inverse tangent function with an appropriate restriction to the tangent function. Let

$$f(x) = \tan x, \qquad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Note that f assumes all real values, as shown in Figure 67. Here, f is increasing, so that it is one-to-one and has an inverse.

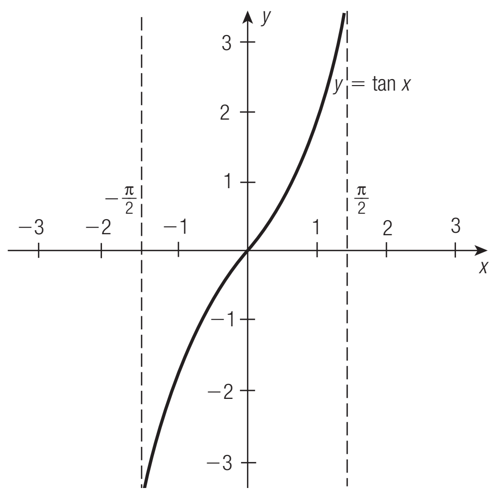


FIGURE 67 Graph of $y = \tan x$, $-\pi/_2 < x < \pi/_2$

The inverse tangent function, denoted by **arctan**, or **tan^{-1}**, is defined by

$$\tan^{-1} x = y \qquad \text{if and only if} \qquad \tan y = x$$

where $-\infty < x < \infty$ and $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$.

Proceeding as before, we sketch the graph of $y = \tan^{-1} x$ in Figure 68.

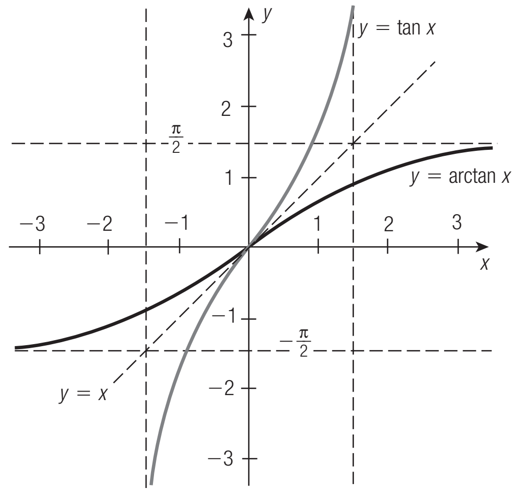


FIGURE 68 Graph of $y = \arctan x$

EXAMPLE 4 APPLYING THE INVERSE TRIGONOMETRIC FUNCTIONS

Find.

(a) $\tan^{-1} \sqrt{3}$ (b) $\tan^{-1}(\cos \pi)$

SOLUTION

(a) If $y = \tan^{-1} \sqrt{3}$, then $\tan y = \sqrt{3}$. Since $-\frac{\pi}{2} < y < \frac{\pi}{2}$, we must have $y = \frac{\pi}{3}$.

(b) Since $\cos \pi = -1$, we let $y = \tan^{-1}(-1)$. Then $\tan y = -1$, where $-\frac{\pi}{2} < y < \frac{\pi}{2}$. Therefore, $y = -\frac{\pi}{4}$.

EXAMPLE 5 APPLYING THE INVERSE TRIGONOMETRIC FUNCTIONS

Find.

$$\sin\left[\arctan\left(-\frac{4}{3}\right)\right]$$

SOLUTION

If we let $t = \arctan\left(-\frac{4}{3}\right)$, then $\tan t = -\frac{4}{3}$ and $-\frac{\pi}{2} < t < 0$. We may think of t as an angle in the fourth quadrant. Draw a right triangle as shown in Figure 69. By the Pythagorean Theorem, the hypotenuse has length 5. Therefore, $\sin t = -\frac{4}{5}$. (Note that $\sin t < 0$ if t is in the fourth quadrant.)

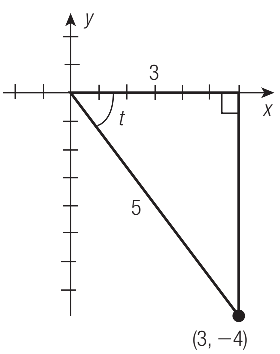


FIGURE 69 Right Triangle Trigonometry

✔ Progress Check

Find

$$\cot\left[\sin^{-1}\left(-\frac{5}{13}\right)\right]$$

Answers

$-\frac{12}{5}$

Graphing Calculator Alert

The values of the inverse trigonometric functions can be found by using a calculator. For example, to find $\sin^{-1} 1$, we must first decide if we want our answer in degrees, radians, or grads. If we place our calculator in the DEGREE mode, for most calculators we press

$$\boxed{1} \quad \boxed{\text{INV}} \quad \boxed{\text{SIN}}$$

and the display reads 90°. If the calculator is set for RADIAN mode and we press

$$\boxed{1} \quad \boxed{\text{INV}} \quad \boxed{\text{SIN}}$$

we obtain 1.5707963, which is $\frac{\pi}{2}$ to seven decimal places. All of the restrictions for the inverse trigonometric functions are built into the calculator. For example, to calculate $\tan^{-1}(-1)$ in radian mode, we press

$$\boxed{-1} \quad \boxed{\text{INV}} \quad \boxed{\text{TAN}}$$

and obtain -0.7853982, which is $-\frac{\pi}{4}$ to seven decimal places. Note that some calculators have $\sin^{-1}x$, $\cos^{-1}x$, and $\tan^{-1}x$ as 2^{nd} or SHIFTed functions. Do *not* use the $\boxed{x^{-1}}$ key to evaluate the inverse trigonometric functions.

✔ Progress Check

Use a calculator to find the following to seven decimal places in radians.

(a) $\sin^{-1}(-0.725)$ (b) $\sec(\arcsin(-0.429))$

Answers

(a) -0.8110344 (b) 1.1070464

EXAMPLE 6 APPLYING THE INVERSE TRIGONOMETRIC FUNCTIONS

Find the solutions of $3 \sin x = 1$ in the interval $[0, \frac{\pi}{2}]$ to seven decimal places.

SOLUTION

First we obtain

$$\sin x = \frac{1}{3}$$

and then

$$x = \sin^{-1} \frac{1}{3} \approx 0.3398369$$

EXAMPLE 7 APPLYING THE INVERSE TRIGONOMETRIC FUNCTIONS

Find the solutions of $2 \sin x = \sqrt{2}$ in the interval $[\frac{\pi}{2}, \pi]$.

SOLUTION

Since

$$\sin x = \frac{\sqrt{2}}{2}$$

we have

$$x = \sin^{-1} \frac{\sqrt{2}}{2} = \frac{\pi}{4}$$

This is *not* the solution since $\frac{\pi}{4}$ does not lie in the interval $[\frac{\pi}{2}, \pi]$, namely, the second quadrant. However, observe that

$$\sin x = \frac{\sqrt{2}}{2}$$

is positive, and $\sin x$ is positive in both the first and second quadrants. Since $\frac{3\pi}{4}$ is in the second quadrant and has reference angle $\frac{\pi}{4}$, we see that $\frac{3\pi}{4}$ is the solution.

EXAMPLE 8 APPLYING THE INVERSE TRIGONOMETRIC FUNCTIONS

Find the solutions of $5 \cos^2 x - 3 = 0$ in the interval $[0, \pi]$.

SOLUTION

We treat the equation as quadratic in cosine.

$$5 \cos^2 x = 3$$

$$\cos x = \pm\sqrt{\frac{3}{5}} = \pm\frac{\sqrt{15}}{5}$$

Thus

$$x = \cos^{-1} \frac{\sqrt{15}}{5} \qquad \text{or} \qquad x = \cos^{-1}\left(-\frac{\sqrt{15}}{5}\right)$$

Since $x = \cos^{-1} \frac{\sqrt{15}}{5} \approx 0.6847192$ and $x = \cos^{-1}\left(-\frac{\sqrt{15}}{5}\right) = 2.4568735$, both values lie in the interval $[0, \pi]$. Therefore, they are both solutions of the given equation.

✔ Progress Check

Find the solutions of $2\sin^2 x + 2\sin x - 1 = 0$ that are in the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Answers

$$\sin^{-1}\left(\frac{-1+\sqrt{3}}{2}\right) \approx 0.3747344$$

Note that $\sin^{-1}\left(\frac{-1-\sqrt{3}}{2}\right)$ does not exist.

WARNING

It is important to remember the domain and range for each inverse trigonometric function. For example, given

$$y = \tan^{-1}(-1)$$

students often write $y = \frac{3\pi}{4}$, which is incorrect. Although $\tan\frac{3\pi}{4} = -1$, the correct answer is $y = -\frac{\pi}{4}$ since the answer must come from the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

Although it is possible to define $\csc^{-1} x$, $\sec^{-1} x$, and $\cot^{-1} x$ by similarly restricting the functions $\csc x$, $\sec x$, and $\cot x$, respectively, so that they are one-to-one, we will not discuss these functions here.

Exercise Set 7.7

In Exercises 1–18, evaluate the given expression.

1. $\sin^{-1}\left(-\frac{1}{2}\right)$
2. $\arccos\frac{\sqrt{3}}{2}$
3. $\arctan\sqrt{3}$
4. $\tan^{-1} 0$
5. $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$
6. $\cos^{-1}(-1)$
7. $\arccos\left(-\frac{\sqrt{3}}{2}\right)$
8. $\tan^{-1}\frac{\sqrt{3}}{3}$
9. $\sin^{-1}(-1)$
10. $\arctan 1$
11. $\cos^{-1} 0$
12. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
13. $\cos^{-1} 1$
14. $\arcsin\frac{\sqrt{2}}{2}$
15. $\arctan(-1)$
16. $\sin^{-1} 0$
17. $\cos^{-1}\left(-\frac{1}{2}\right)$
18. $\arcsin\frac{1}{2}$

In Exercises 19–24, evaluate the given expression using a calculator that has a key marked $\boxed{\text{INV}}$ or an equivalent notation.

19. $\sin^{-1} 0.3709$
20. $\arctan 1.398$
21. $\cos^{-1}(-0.7648)$
22. $\tan^{-1}(-3.010)$
23. $\arcsin 0.9636$
24. $\arccos(-0.921)$

In Exercises 25–36, evaluate the given expression without using tables or a calculator.

25. $\sin(\arctan 1)$
26. $\cos\left[\arcsin\left(-\frac{1}{2}\right)\right]$
27. $\tan^{-1}\left(\cos\frac{\pi}{2}\right)$
28. $\sin^{-1}(\sin 0.62)$

29. $\cos^{-1}\left(\sin\dfrac{9\pi}{4}\right)$

30. $\tan(\sin^{-1} 0)$

31. $\cos^{-1}\left(\cos\dfrac{2\pi}{3}\right)$

32. $\sin^{-1}\left(\cos\dfrac{\pi}{6}\right)$

33. $\tan\left[\sin^{-1}\left(-\dfrac{5}{13}\right)\right]$

34. $\sin\left[\arctan\left(-\dfrac{12}{5}\right)\right]$

35. $\cos\left[\sin^{-1}\left(\dfrac{4}{5}\right)\right]$

36. $\tan\left[\cos^{-1}\left(-\dfrac{3}{5}\right)\right]$

In Exercises 37–44, solve the given equation.

37. $\sin x = \dfrac{\sqrt{3}}{2},\qquad x \in \left[0, \dfrac{\pi}{2}\right]$

38. $\cos x = -\dfrac{1}{2},\qquad x \in [0, \pi]$

39. $\tan x = \dfrac{1}{\sqrt{3}},\qquad x \in \left[0, \dfrac{\pi}{2}\right]$

40. $2\tan x = 2,\qquad x \in [0, 2\pi]$

41. $2\sin x = -1,\qquad x \in [\pi, 2\pi]$

42. $\cos x = \dfrac{\sqrt{3}}{2},\qquad x \in [\pi, 3\pi]$

43. $\tan x = \sqrt{3},\qquad x \in \left[\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right]$

44. $\tan x = -1,\qquad x \in [-\pi, 2\pi]$

In Exercises 45–50, use the inverse trigonometric functions to express the solutions of the given equation exactly.

45. $7\sin^2 x - 1 = 0,\qquad x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

46. $6\cos^2 y - 5 = 0,\qquad y \in [0, \pi]$

47. $12\cos^2 x - \cos x - 1 = 0,\qquad x \in [0, \pi]$

48. $2\tan^2 t + 4\tan t - 3 = 0,\qquad t \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

49. $9\sin^2 t - 12\sin t + 4 = 0,\qquad t \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

50. $3\cos^2 x - 7\cos x - 6 = 0,\qquad x \in [0, \pi]$

In Exercises 51–54, provide a value for x to show that the equation is not an identity.

51. $\sin^{-1} x = \dfrac{1}{\sin x}$

52. $(\sin^{-1} x)^2 + (\cos^{-1} x)^2 = 1$

53. $\sin^{-1}(\sin x) = x$

54. $\arccos(\cos x) = x$

In Exercises 55–58, use a calculator to assist in finding all solutions in the indicated interval.

55. $2\cos^2 x + \cos x = 2,\qquad [0, \pi]$

56. $2\tan^2 x = 3,\qquad \left[0, \dfrac{\pi}{2}\right]$

57. $\sin^2 x - 2\sin x - 2 = 0,\quad \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

58. $9\cos^2 x + 3\cos x = 2,\qquad [0, \pi]$

Exercises 59–61 refer to the Graphing Calculator Power User's Corner on "Graphing Inverse Functions" in Section 3.5. Modify that example to graph each of the following trigonometric functions, its inverse, and $y = x$ in the given viewing rectangle.

59. Set $-2 \le X \le 2$, $-2 \le Y \le 2$, $-1.57 \le T \le 1.57$, and GRAPH $y = \sin x$ and its inverse.

60. Set $-2 \le X \le 3.5$, $-2 \le Y \le 3.5$, $0 \le T \le 3.14$, and GRAPH $y = \cos x$ and its inverse.

61. Set $-5 \le X \le 5$, $-5 \le Y \le 5$, $-1.57 \le T \le 1.57$, and GRAPH $y = \tan x$ and its inverse.

7.8 Applications

In Section 7.2, we introduced the expression "to solve a triangle," meaning to find the length of each side and the measure of each angle. In this section, we return to using the units of degrees to measure the size of the angles.

Many applied problems involve right triangles. We are now prepared to use our ability in solving triangles to tackle a variety of problems.

EXAMPLE 1 SOLVING A TRIANGLE

In triangle ABC, $\gamma = 90°$, $a = 12.8$, and $b = 22.5$. Find approximate values for the remaining parts of the triangle.

SOLUTION

The parts of the triangle are displayed in Figure 70. Using angle β, we have

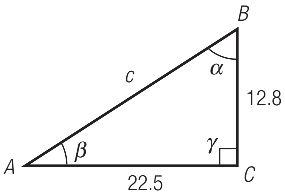


FIGURE 70 Diagram for Example 1

$$\tan \beta = \frac{12.8}{22.5} \approx 0.5689$$

Solving for β,

$$\beta = \tan^{-1} 0.5689 \approx 29.64°$$

Sometimes, we want the answer in degrees and minutes.

$$(0.64°)(60') \approx 38'$$

Therefore, $\beta = 29°38'$. Since $\alpha + \beta + \gamma = 180°$, $\alpha \approx 60°22'$.

We can find c using the Pythagorean Theorem.

$$c = \sqrt{a^2 + b^2} = \sqrt{(12.8)^2 + (22.5)^2}$$
$$= \sqrt{163.84 + 506.25} = \sqrt{670.09} \approx 25.9$$

(Verify these values by finding $\sin \alpha$ two ways.)

✔ Progress Check

In triangle ABC, $\gamma = 90°$, $a = 17.4$ and $b = 38.2$. Solve the triangle.

Answers

$\alpha \approx 24°29'$ $\quad\beta \approx 65°31'$ $\quad c \approx 42.0$

EXAMPLE 2 APPLYING TRIGONOMETRY

A ladder leaning against a building makes an angle of 35° with the ground. If the bottom of the ladder is 5 meters from the building, how long is the ladder? To what height does it rise along the building?

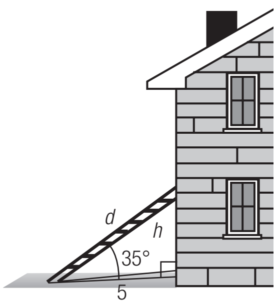


FIGURE 71 Diagram for Example 2

SOLUTION

In Figure 71, we seek the length d of the ladder and the height h along the building. Using right triangle trigonometry,

$$\cos 35° = \frac{5}{d} \qquad \text{and} \qquad \tan 35° = \frac{h}{5}$$

$$d = \frac{5}{\cos 35°} \qquad\qquad h = 5 \tan 35°$$

$$d \approx \frac{5}{0.8192} \qquad\qquad h \approx 5(0.7002)$$

$$d \approx 6.1 \text{ meters} \qquad \text{and} \qquad h \approx 3.5 \text{ meters}$$

Elevation and Depression

There are two terms that will occur frequently in our word problems. The **angle of elevation** is the angle between the horizontal and the line of sight when looking up. In Figure 72(a), θ is the angle of elevation of the top T of a tree from a point x meters from the base of the tree.

✔ Progress Check

The string of a kite makes an angle of 32°30′ with the ground. If 125 meters of string have been let out, how high is the kite?

Answers

Approximately 67 meters

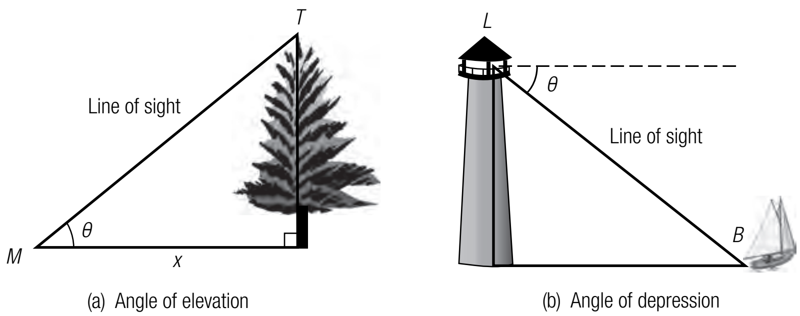


FIGURE 72 Angles of Elevation and Depression

The **angle of depression** is the angle between the horizontal and the line of sight when looking down. In Figure 72(b), θ is the angle of depression of a boat B as seen from a lighthouse L.

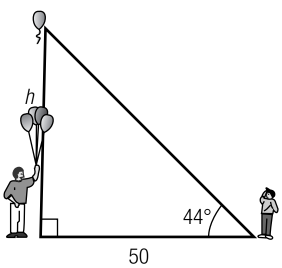


FIGURE 73 Diagram for Example 3

EXAMPLE 3 ANGLE OF ELEVATION

A vendor of balloons inadvertently releases a balloon, which rises straight up. A child standing 50 feet from the vendor watches the balloon rise. When the angle of elevation of the balloon reaches 44°, how high is the balloon?

SOLUTION

We seek the height h in Figure 73. Thus,

$$\tan 44° = \frac{h}{50}$$

$$h = 50 \tan 44°$$

$$h \approx 50(0.9657) \approx 48$$

The balloon has risen approximately 48 feet.

EXAMPLE 4 ANGLE OF DEPRESSION

A forest ranger is in a tower 65 feet above the ground. If the ranger spots a fire at an angle of depression of $6°30'$, how far is the fire from the base of the tower, assuming level terrain?

SOLUTION

We need to find the distance d in Figure 74. Since $\theta + 6°30' = 90°$, $\theta = 83°30'$.

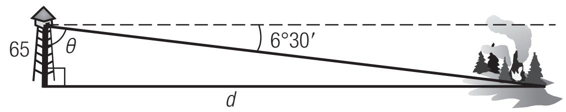


FIGURE 74 Diagram for Example 4

Then

$$\tan\theta = \frac{d}{65}$$
$$d = 65 \tan 83°30'$$
$$= 65 \tan 83.5°$$
$$d \approx 65(8.7769) \approx 570$$

The fire is approximately 570 feet from the base of the tower.

EXAMPLE 5 APPLYING TRIGONOMETRY

A mathematics professor walks toward the university clock tower on the way to her office and decides to find the height of the clock above ground. She determines the angle of elevation to be 30° and, after proceeding an additional 60 feet toward the base of the tower, finds the angle of elevation to be 40°. What is the height of the clock?

SOLUTION

In Figure 75, we seek to determine h. From triangle ACD,

$$\tan 30° = \frac{h}{d + 60} \qquad \text{or} \qquad h = (d + 60)(\tan 30°)$$

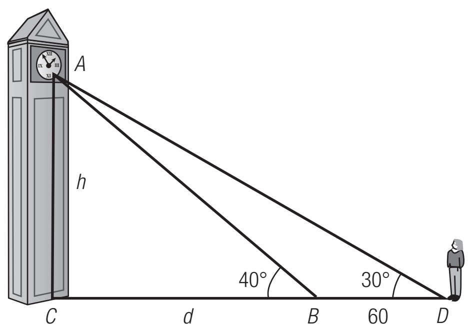


FIGURE 75 Diagram for Example 5

From triangle ACB,

$$\tan 40° = \frac{h}{d} \qquad \text{or} \qquad d = \frac{h}{\tan 40°}$$

Substituting for d in the first expression, we obtain

$$\begin{aligned} h &= \left(\frac{h}{\tan 40°} + 60\right) \tan 30° \\ &\approx 0.6881\, h + 34.6410 \\ 0.3119\, h &\approx 34.641 \\ h &\approx 111.1 \end{aligned}$$

The height of the clock is approximately 111.1 feet.

Navigation and Surveying

In navigation and surveying, directions are often given by **bearings**, which specify an acute angle and its direction from the north-south line. In Figure 76(a), the bearing of point B from point A is N 40° E, that is, 40° east of north. In Figure 76(b), the bearing of point B from point A is N 30° W. In Figures 76(c) and 76(d), it is S 60° W and S 20° E, respectively.

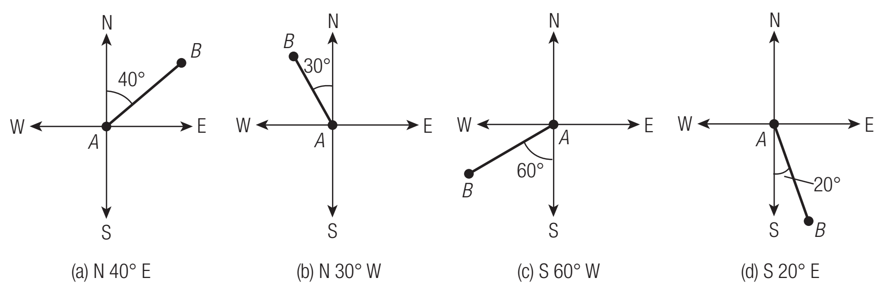


FIGURE 76 Bearings

EXAMPLE 6 BEARINGS

A ship leaves port at 10 A.M. and heads due east at a rate of 22 mph. At 11 A.M. the course is changed to S 52° E. Find the distance and bearing of the ship from the point of departure at noon.

SOLUTION

The ship travels due east from port, point A, reaches B at 11 A.M. and changes direction, arriving at E at noon, as shown in Figure 77. Since the ship travels at 22 mph, $\overline{AB}$ and $\overline{BE}$ are each 22 miles in length. Further, since angle EBS has a measure of 52°, we find angle $\beta = 38°$. From right triangle BCE,

$$\cos \beta = \frac{e}{22} \qquad \text{or} \qquad e = 22 \cos 38° \approx 17.3 \text{ miles}$$

$$\sin \beta = \frac{b}{22} \qquad \text{or} \qquad b = 22 \sin 38° \approx 13.5 \text{ miles}$$

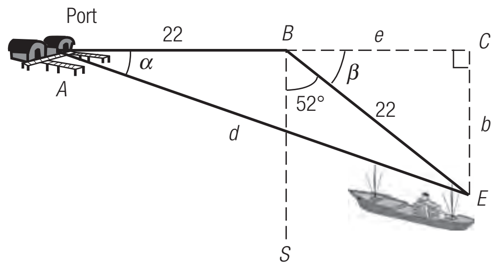


FIGURE 77 Diagram for Example 6

We now know two sides of right triangle ACE, namely

$$\overline{AC} = 22 + e \approx 22 + 17.3 = 39.3$$
$$\overline{CE} = b \approx 13.5$$

From the Pythagorean Theorem,

$$d = \sqrt{(\overline{AC})^2 + (\overline{CE})^2} \approx \sqrt{(39.3)^2 + (13.5)^2} = \sqrt{1726.74} \approx 41.6$$

To find the bearing,

$$\tan \alpha = \frac{\overline{CE}}{\overline{AC}} \approx \frac{13.5}{39.3} \approx 0.3435$$

so

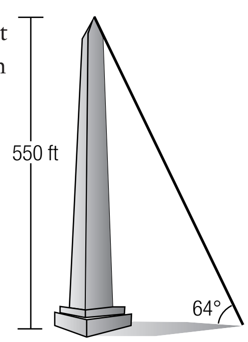
$$\alpha = 19°$$

Therefore, the ship is approximately 41.6 miles from port at a bearing of approximately S 71° E.

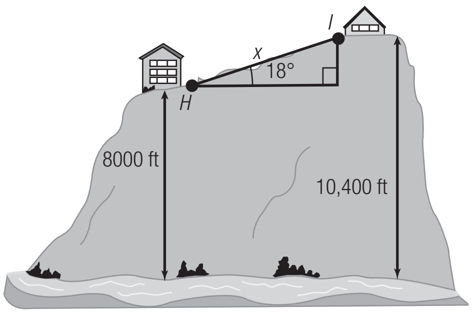
Exercise Set 7.8

In Exercises 1 and 2, find the required part of triangle ABC if $\gamma = 90°$.

1. If $a = 12$ and $b = 16$, find α.
2. If $a = 5$ and $b = 15$, find β.
3. A ladder 20 feet in length touches a wall at a point 16 feet above the ground. Find the angle the ladder makes with the ground.
4. A monument is 550 feet high. What is the length of the shadow cast by the monument when the sun is 64° above the horizon?

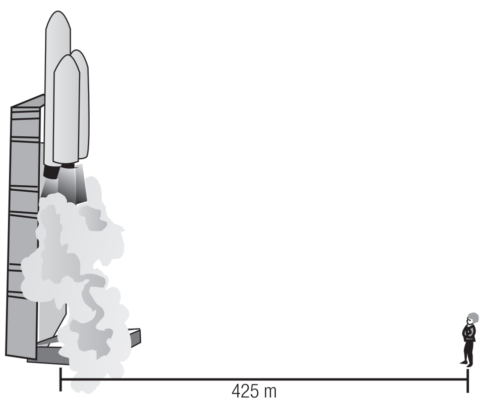


5. Find the angle of elevation of the sun when a tower 45 meters in height casts a horizontal shadow 25 meters in length.
6. A technician positioned on an oil-drilling rig 120 feet above the water spots a boat at an angle of depression of 16°. How far is the boat from the rig?
7. A mountainside hotel is located 8000 feet above sea level. From the hotel, a trail leads farther up the mountain to an inn at an elevation of 10,400 feet. If the trail has an angle of inclination of 18° (that is, the angle of elevation of the inn from the hotel is 18°), find the distance along the trail from the hotel to the inn.



8. A hill is known to be 200 meters high. A surveyor standing on the ground finds the angle of elevation of the top of the hill to be 42°50′. Find the distance from the surveyor's feet to a point directly below the top of the hill.

9. An observer is 425 meters from a launching pad when a rocket is launched vertically. If the angle of elevation of the rocket at its apogee (highest point) is 66°20′, how high does the rocket rise?

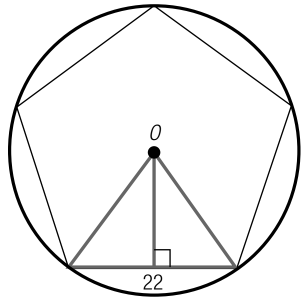


10. An airplane pilot wants to climb from an altitude of 6000 feet to an altitude of 16,000 feet. If the plane climbs at an angle of 9° with a constant speed of 22,000 feet per minute, how long will it take to reach the increased altitude?

11. A rectangle is 16 inches long and 13 inches wide. Find the measure of the angles formed by a diagonal with the sides.

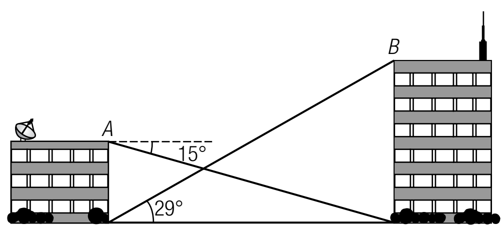
12. The sides of an isosceles triangle are 15, 15, and 26 centimeters. Find the measures of the angles of the triangle. (*Hint:* The altitude of an isosceles triangle bisects the base.)

13. The side of a regular pentagon is 22 centimeters. Find the radius of the circle circumscribed about the pentagon. (*Hint:* The radii from the center of the circumscribed circle to any two adjacent vertices of the regular pentagon form an isosceles triangle. The altitude of an isosceles triangle bisects the base.)



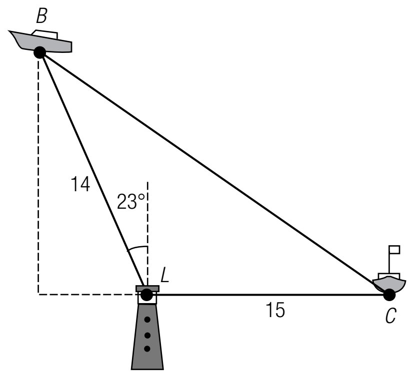
14. To determine the width of a river, markers are placed at each side of the river in line with the base of a tower that rises 23.4 meters above the ground. From the top of the tower, the angles of depression of the markers are 58°20′ and 11°40′. Find the width of the river.

15. The angle of elevation of the top of building *B* from the base of building *A* is 29°. From the top of building *A*, the angle of depression of the base of building *B* is 15°. If building *B* is 110 feet high, find the height of building *A*.



16. A ship leaves port at 2 P.M. and heads due east at a rate of 40 kilometers per hour. At 4 P.M. the course is changed to N 32° E. Find the distance and bearing of the ship from the point of departure at 6 P.M.

17. An attendant in a lighthouse receives a request for aid from a stalled craft located 15 miles due east of the lighthouse. The attendant contacts a second boat located 14 miles from the lighthouse at a bearing of N 23° W. What is the distance of the rescue ship from the stalled craft?



Chapter Summary

Terms and Symbols

Key Ideas for Review

Topic | **Page** | **Key Idea**

Measurement of Angles 441

Angles are usually measured in degrees or radians. A circle has 360° or 2π radians.

Conversion Formulas 442

Since

$$\pi \text{ radians} = 180°$$

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^{\circ} \qquad \text{and} \qquad 1° = \left(\frac{\pi}{180}\right) \text{radians}$$

Reference Angle 446

The reference angle θ' associated with the angle θ is the acute angle formed by the terminal side of θ and the x-axis.

Right Triangle Trigonometry 451

Right triangle trigonometry relates a trigonometric function of an angle θ of a right triangle to the ratio of the lengths of two of its three sides: the *hypotenuse*, *opposite*, and *adjacent*.

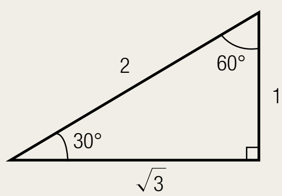
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$

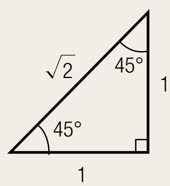
$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\tan\theta = \frac{\text{opposite}}{\text{adjacent}} \qquad \cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

"Special" Right Triangles 444

Two triangles that can be helpful in determining special values of trigonometric functions are





Points on the Unit Circle 464

For every real number t, there is a unique point $P(t)$ on the unit circle. If the rectangular coordinates of $P(t)$ are (x, y), we write

$$P(t) = P(x, y)$$

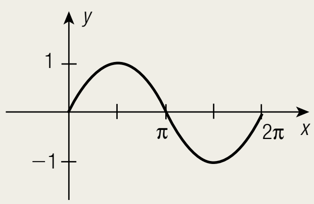
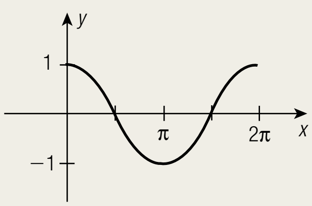
We defined $\cos t = x$ and $\sin t = y$ so that

$$P(t) = P(\cos t, \sin t)$$

Furthermore,

$$P(t) = P(t + 2\pi n)$$

where n is any integer.

Topic	Page	Key Idea
Points Not at the Origin	476	If P has coordinates (x, y) with $r = \sqrt{x^2 + y^2} \neq 0$, then $$P(x, y) = P(r \cos \theta, r \sin \theta)$$ and $$\cos \theta = \frac{x}{r}, \quad \sin \theta = \frac{y}{r}$$
Trigonometric or Circular Functions	463	If $P(t)$ is a point on the unit circle, then $$\tan t = \frac{\sin t}{\cos t} \qquad \cot t = \frac{\cos t}{\sin t}$$ $$\sec t = \frac{1}{\cos t} \qquad \csc t = \frac{1}{\sin t}$$
Periodicity	488	All trigonometric functions are periodic. Both $\sin t$ and $\cos t$ have period 2π, whereas $\tan t$ has period π.
Signs	480	Of $\sin t$, $\cos t$, and $\tan t$, all are positive in quadrant I, $\sin t$ is positive in quadrant II, $\tan t$ is positive in quadrant III, and $\cos t$ is positive in quadrant IV.
Even and Odd Functions	482	Sine and tangent are odd functions, whereas cosine is an even function. $$\sin(-t) = -\sin t$$ $$\cos(-t) = \cos t$$ $$\tan(-t) = \tan t$$
Identity	482	$$\sin^2 t + \cos^2 t = 1$$
Graphs of Sine and Cosine	489	The graph of $y = \sin x$ for $0 \leq x \leq 2\pi$ is  The graph of $y = \cos x$ for $0 \leq x \leq 2\pi$ is 

Key Ideas for Review

Topic	Page	Key Idea
Amplitude, Period, and Phase Shift	496	To sketch the graph of $f(x) = A \sin(Bx + C)$ or $f(x) = A \cos(Bx + C)$ for $A \neq 0$, $B \neq 0$, note that 1. The amplitude is $\lvert A \rvert$ 2. The period is $\frac{2\pi}{\lvert B \rvert}$. 3. The phase shift is $-\frac{C}{B}$.
Inverse Trigonometric Functions	504	To define inverse trigonometric functions, it is necessary to restrict the domain of the trionometric functions so that the resulting function is one-to-one.

Review Exercises

Solutions to exercises whose numbers are in **blue** are in the Solutions section in the back of the book.

In Exercises 1–4, convert from degree measure to radian measure or from radian measure to degree measure.

1. $-60°$
2. $\dfrac{3\pi}{2}$
3. $-\dfrac{5\pi}{12}$
4. $45°$

In Exercises 5–7, determine if the pair of angles are coterminal.

5. $100°, \dfrac{5\pi}{9}$
6. $\dfrac{4\pi}{3}, 480°$
7. $\dfrac{5\pi}{4}, -135°$

In Exercises, 8–11, determine the quadrant in which t or θ lies.

8. $t = \dfrac{11\pi}{6}$
9. $\theta = -220°$
10. $\theta = 490°$
11. $t = -\dfrac{11\pi}{3}$

In Exercises 12–14, find the reference angle of the given angle.

12. $310°$
13. $-185°$
14. $405°$

15. If a central angle θ subtends an arc of length 14 centimeters on a circle whose radius is 10 centimeters, find the radian measure of θ.
16. A central angle of $\frac{2\pi}{3}$ radians subtends an arc of length $\frac{5\pi}{2}$ centimeters. Find the radius of the circle.

In Exercises 17–20, replace each given real number t by t', $0 \leq t' < 2\pi$, so that $P(t') = P(t)$.

17. $\dfrac{9\pi}{2}$
18. $-\dfrac{15\pi}{2}$
19. -6π
20. $\dfrac{23\pi}{3}$

In Exercises 21–23, express the required trigonometric function as a ratio of the given parts of the right triangle ABC with $\gamma = 90°$.

21. If $a = 5$ and $b = 12$, find $\sin \alpha$.
22. If $a = 3$ and $c = 5$, find $\tan \beta$.
23. If $a = 4$ and $b = 7$, find $\sec \alpha$.

In Exercises 24–27, the point P lies on the terminal side of the angle θ. Find the value of the required trigonometric function without using tables or a calculator.

24. $P(-\sqrt{3}, 1)$, $\csc \theta$
25. $P(\sqrt{2}, -\sqrt{2})$, $\cot \theta$

26. $P(-1, -\sqrt{3})$, $\cos\theta$ 27. $P(\sqrt{2}, \sqrt{2})$, $\sin\theta$

In Exercises 28–32, find the rectangular coordinates of the given point without using tables or a calculator.

28. $P\left(\frac{7\pi}{6}\right)$ 29. $P\left(-\frac{8\pi}{3}\right)$

30. $P\left(\frac{5\pi}{6}\right)$ 31. $P\left(-\frac{7\pi}{4}\right)$

32. $P\left(\frac{11\pi}{6}\right)$

In Exercises 33–36, determine the value of the indicated trigonometric function without using tables or a calculator.

33. $\sin\frac{2\pi}{3}$ 34. $\sec\left(-\frac{5\pi}{4}\right)$

35. $\tan\frac{5\pi}{6}$ 36. $\csc\left(-\frac{\pi}{6}\right)$

In Exercises 37–41, $P(t) = (\frac{4}{5}, -\frac{3}{5})$. Use the symmetries of the circle to find the rectangular coordinates of the given point.

37. $P(t - \pi)$ 38. $P\left(t + \frac{\pi}{2}\right)$

39. $P(-t)$ 40. $P\left(t - \frac{\pi}{2}\right)$

41. $P(-t - \pi)$

In Exercises 42–45, find a value of t in the interval $[0, 2\pi)$ satisfying the given conditions.

42. $\sin t = -\frac{\sqrt{2}}{2}$, $P(t)$ in quadrant III

43. $\cos t = \frac{\sqrt{3}}{2}$, $P(t)$ in quadrant IV

44. $\cot t = \frac{\sqrt{3}}{3}$, $P(t)$ in quadrant I

45. $\sec t = -2$, $P(t)$ in quadrant II

In Exercises 46 and 47, find the quadrant in which t lies if the following conditions hold.

46. $\sin t < 0$ and $\cos t > 0$

47. $\sin(-t) > 0$ and $\tan t > 0$

In Exercises 48–51, use the trigonometric identities

$$\sin^2 t + \cos^2 t = 1 \qquad \tan t = \frac{\sin t}{\cos t}$$

to find the indicated value under the given conditions.

48. If $\cos t = \frac{3}{5}$ and $P(t)$ is in quadrant IV, find $\cot t$.

49. If $\sin t = -\frac{4}{5}$ and $\tan t > 0$, find $\sec t$.

50. If $\sin t = \frac{12}{13}$ and $\cos t < 0$, find $\tan t$.

51. If $\cos t = -\frac{5}{13}$ and $\tan t < 0$, find $\csc t$.

In Exercises 52 and 53, use the trigonometric identities to transform the first expression into the second.

52. $(\sin t)(\sec t)$, $\tan t$ 53. $\frac{\sin t}{\cos^2 t}$, $(\tan t)(\sec t)$

In Exercises 54 and 55, use a calculator to evaluate the given expression.

54. $\cos 3.71 - \sin 1.44$ 55. $\tan(-2.74)$

In Exercises 56 and 57, sketch the graph of the given function.

56. $f(x) = 1 - \sin x$ 57. $f(x) = 2\sin\left(\frac{x}{2} + \pi\right)$

In Exercises 58–60, determine the amplitude, period and phase shift for each given function. Sketch the graph. Then, determine appropriate RANGE values and GRAPH the functions on your graphing calculator. Be sure your calculator is in RADIAN mode.

58. $f(x) = -\cos(2x - \pi)$

59. $f(x) = 4\sin\left(-x + \frac{\pi}{2}\right)$

60. $f(x) = -2\sin\left(\frac{x}{3} + \frac{\pi}{3}\right)$

In Exercises 61–63, evaluate the given expression.

61. $\arcsin\left(-\frac{1}{2}\right)$ 62. $\tan(\cos^{-1} 1)$

63. $\tan(\tan^{-1} 5)$

64. Use the inverse cosine function to express the exact solutions of the equation

$$5\cos^2 x - 4 = 0$$

Review Exercises

In Exercises 65-68, use a calculator to find the required part of triangle ABC with $\gamma = 90°$.

65. If $a = 50$ and $b = 60$, find α.

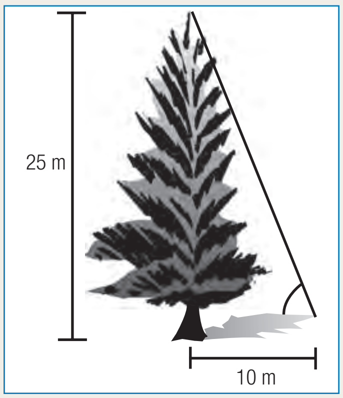
66. If $a = 40$ and $\beta = 20°$, find b.

67. If $a = 20$ and $\alpha = 52°$, find c.

68. If $b = 15$ and $\alpha = 25°$, find c.

69. A ladder 6 meters in length leans against a vertical wall. If the ladder makes an angle of 65° with the ground, find the height that the ladder reaches above the ground.

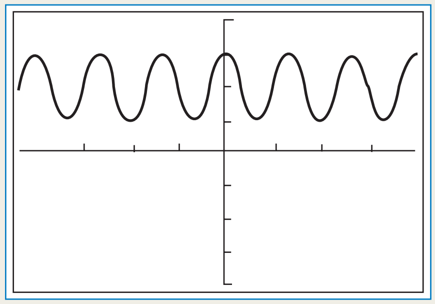
70. Find the angle of elevation of the sun when a tree 25 meters in height casts a horizontal shadow 10 meters in length.



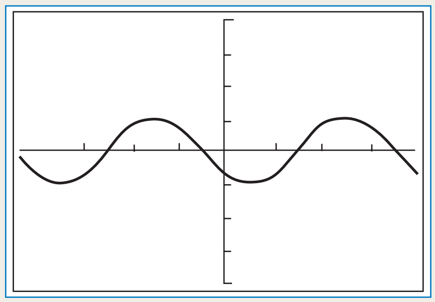
71. A rectangle is 22 centimeters long and 16 centimeters wide. Find the measure of the smaller angle formed by the diagonal with a side.

In Exercises 72 and 73, the following graphs are variations of the function $y = \cos x$. Determine A, B, and C, where $y = A \cos (Bx + C)$ for each graph below. Verify your answer by GRAPHing the function on your graphing calculator. (All graphs are drawn in the viewing rectangle $-6.28 \le X \le 6.28$, $-4 \le Y \le 4$, with XSCL $= 1.57$ and YSCL $= 1$.)

72.



73.



Review Test

In Exercises 1–3, convert from degree measure to radian measure or from radian measure to degree measure.

1. $\dfrac{5\pi}{3}$ 2. $-200°$ 3. $75°$

In Exercises 4 and 5, find the angle θ, $0° \le \theta < 360°$, that is coterminal with the given angle.

4. $-25°$ 5. $\dfrac{17\pi}{4}$

In Exercises 6 and 7, find the reference angle of the given angle.

6. $160°$ 7. $\dfrac{7\pi}{4}$

8. If a central angle θ subtends an arc of length 12 inches on a circle whose radius is 15 inches, find the radian measure of θ.

In Exercises 9 and 10, replace the given real number t by t', $0 \le t' < 2\pi$, so that $P(t') = P(t)$.

9. $\dfrac{19\pi}{3}$ 10. -22π

In Exercises 11 and 12, ABC is a right triangle with $\gamma = 90°$. Express the required trigonometric function as a ratio of the given parts of the triangle.

11. If $a = 7$ and $b = 5$, find $\tan \alpha$.
12. If $b = 5$ and $c = 15$, find $\sec \alpha$.

In Exercises 13–15, the point P lies on the terminal side of the angle θ. Find the value of the required trigonometric function without using tables or a calculator.

13. $P(-\sqrt{2}, \sqrt{2})$, $\cot \theta$ 14. $P(0, -5)$, $\sin \theta$
15. $P(2, 2\sqrt{3})$, $\sec \theta$

In Exercises 16 and 17, find the rectangular coordinates of the given point.

16. $P\left(\dfrac{29\pi}{6}\right)$ 17. $P\left(-\dfrac{\pi}{3}\right)$

In Exercises 18–20, $P(t) = \left(-\frac{5}{13}, \frac{12}{13}\right)$. Use the symmetries of the circle to find the rectangular coordinates of the given point.

18. $P(t + \pi)$ 19. $P\left(t - \dfrac{\pi}{2}\right)$
20. $P(-t)$

In Exercises 21 and 22, determine the value of the indicated trigonometric function without using tables or a calculator.

21. $\cos \dfrac{7\pi}{3}$ 22. $\csc\left(-\dfrac{2\pi}{3}\right)$

In Exercises 23 and 24, find a value of t in the interval $[0, 2\pi)$ satisfying the given conditions.

23. $\tan t = 1$, $P(t)$ in quadrant III
24. $\sec t = \sqrt{2}$, $P(t)$ in quadrant IV

In Exercises 25 and 26, use the trigonometric identities

$$\sin^2 t + \cos^2 t = 1 \qquad \tan t = \frac{\sin t}{\cos t}$$

to find the indicated value under the given conditions.

25. If $\cos t = -\frac{12}{13}$ and $\tan t > 0$, find $\sin t$.
26. If $\sin t = \frac{3}{5}$ and $P(t)$ is in quadrant II, find $\sec t$.
27. Use the trigonometric identities given for Exercises 25 and 26 to transform

$$1 - \tan x \quad \text{to} \quad \frac{\cos x - \sin x}{\cos x}$$

In Exercises 28 and 29, use a calculator to evaluate the given expression.

28. $\tan(-3.68)$ 29. $\cos 1.15 - \sin 0.72$
30. Sketch the graph of the function f defined by

$$f(x) = x + \cos x$$

In Exercises 31 and 32, determine the amplitude, period, and phase shift of each given function.

31. $f(x) = -2\cos(\pi - x)$ 32. $f(x) = 2\sin\left(\dfrac{x}{2} - \dfrac{\pi}{2}\right)$

Review Test

In Exercises 33 and 34, evaluate the given expression.

33. $\tan^{-1}(-\sqrt{3})$

34. $\cos\left(\sin^{-1}\dfrac{\sqrt{3}}{2}\right)$

35. Use the inverse tangent function to express the exact solutions of the equation

$$6\tan^2 x - 13\tan x + 6 = 0$$

where $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

In Exercises 36–38, use a calculator to find the required part of triangle ABC with $\gamma = 90°$.

36. If $a = 25$ and $c = 30$, find α.

37. If $b = 20$ and $\alpha = 32°$, find c.

38. If $a = 15$ and $b = 20$, find β.

39. From the top of a cliff 100 meters in height, the angle of depression of the entrance to a castle is 36°. Find the distance of the castle from the base of the cliff.